Available online at www.sciencedirect.com
SCIENCE DIRECT

J. Math. Anal. Appl. 308 (2005) 303–313

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Sharp power mean bounds for the Gaussian hypergeometric function

Kendall C. Richards [1]

Department of Mathematics, Southwestern University, Georgetown, TX 78626, USA

Received 27 September 2004

Available online 16 February 2005

Submitted by B.C. Berndt

Abstract

Sharp inequalities are established between the Gaussian hypergeometric function and the power mean. These results extend known inequalities involving the complete elliptic integral and the hypergeometric mean.

Keywords: Hypergeometric function; Elliptic integral; Power mean

1. Introduction

The Gaussian hypergeometric function is given by

$$_2F_1(\alpha, \beta; \gamma : r) \equiv \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} r^k,$$

where $|r| < 1$ and $(\alpha)_k$ is the Pochhammer symbol defined by $(\alpha)_0 = 1$, $(\alpha)_1 = \alpha$, and $(\alpha)_{k+1} = (\alpha)_k(\alpha + k)$ for $k \in \mathbb{N}$ (see [1, p. 556]). Inequalities relating the Gaussian hyper-

E-mail address: richards@southwestern.edu.
[1] The author wishes to extend his gratitude to Clyde Richards and Roger Barnard for useful conversations. Dedicated to Kelly, Bryan, and David.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.01.018

geometric function to various means have been widely studied (e.g., see [5,6,10–12]). In particular, bounds for the complete elliptic integral (see [13, p. 909]),

$$E(r) \equiv \int_0^{\pi/2} (1-r^2\sin^2\theta)^{1/2}\,d\theta = \frac{\pi}{2}\cdot {}_2F_1(-1/2,1/2;1:r^2),$$

in terms of the *power mean of order* λ, given by

$$M_\lambda(\omega,r) \equiv \left[(1-\omega)+\omega(1-r)^\lambda\right]^{1/\lambda},$$

are discussed in [2,4,7,8,15]. This paper provides a generalization of the main result in [8] (motivated by a conjecture of M. Vuorinen in [15]) that

$$E(r) \geqslant \frac{\pi}{2}\left[M_\lambda\left(\frac{1}{2},r^2\right)\right]^{1/2}, \quad r^2\in(0,1), \tag{1}$$

for all λ not exceeding the sharp value of 3/4. The central result of this paper also provides refinements of certain inequalities found in [6,12] involving the *hypergeometric mean* $[{}_2F_1(-a,b;c:r)]^{1/a}$, with $c \geqslant b > 0$. Carlson [12] used an elegant argument involving Euler's integral representation for the hypergeometric function to show that, if $1 \geqslant a$ and $c \geqslant b > 0$, then

$$\left[{}_2F_1(-a,b;c:r)\right]^{1/a} \geqslant M_a\left(\frac{b}{c},r\right), \quad \text{for all } r\in(0,1) \tag{2}$$

(and that the inequality in (2) reverses when $a > 1$). A refinement of (2) was established in [6] where it was also conjectured that if $1 > a > 0$, $b > 0$, $a+b \geqslant 1/2$, and $\lambda \leqslant (a+2b)/(1+2b)$, then

$$\left[{}_2F_1(-a,b;2b:r)\right]^{1/a} \geqslant M_\lambda\left(\frac{1}{2},r\right), \quad \text{for all } r\in(0,1). \tag{3}$$

To reveal natural threshold relationships among the parameters in this context, it is instructive to note that if $a=1$; or $c=\max\{-a,b\}$; or $c=1-2a=2b$; then

$$\left[{}_2F_1(-a,b;c:r)\right]^{1/a} = M_\lambda\left(\frac{b}{c},r\right), \quad \text{when } \lambda=(a+c)/(1+c). \tag{4}$$

If $a=1$, then both sides of (4) reduce to the value $1-br/c$. Since

$$(1-r)^a = \sum_{n=0}^{\infty}\frac{(-a)_n}{n!}r^n = {}_2F_1(-a,b;b:r),$$

Eq. (4) easily follows when $b=c$. That (4) holds when $c=1-2a=2b$ follows from the classical relation

$${}_2F_1(-a,b;2b:r) = \left(\frac{1+\sqrt{1-r}}{2}\right)^{2a} {}_2F_1(-a,1/2-a-b;1/2+b:\xi^2),$$

where $\xi = (1-\sqrt{1-r})/(1+\sqrt{1-r})$ (see [5, p. 132]). In the case that $c=-a$, M_0 is interpreted as a limit (see [9,12]) and Eq. (4) is verified as follows:

$$M_0\left(\frac{b}{c},r\right) \equiv \lim_{\lambda\to 0} M_\lambda\left(\frac{b}{c},r\right) = (1-r)^{-b/c} = \left[{}_2F_1(c,b;c:r)\right]^{-1/c}.$$

In summary, Theorem 1 confirms (3) and hence generalizes (1). Since $\lambda \mapsto M_\lambda$ is increasing and $(a+c)/(1+c) > a$ when $1 > a$, it becomes clear that Theorem 1 also provides a sharp refinement of (2).

2. Main results

Theorem 1. *Suppose* $b > 0$, $1 \geqslant a$, *and* $c \geqslant \max\{-a, b\}$. *If* $c \geqslant \max\{1-2a, 2b\}$, *then*

$$\left[{}_2F_1(-a, b; c : r)\right]^{1/a} \geqslant M_\lambda\left(\frac{b}{c}, r\right), \quad \text{for all } r \in (0,1), \tag{5}$$

if and only if $\lambda \leqslant (a+c)/(1+c)$. *If* $c \leqslant \min\{1-2a, 2b\}$, *then*

$$\left[{}_2F_1(-a, b; c : r)\right]^{1/a} \leqslant M_\lambda\left(\frac{b}{c}, r\right), \quad \text{for all } r \in (0,1), \tag{6}$$

if and only if $\lambda \geqslant (a+c)/(1+c)$. *If* $c = 1-2a = 2b$; *or* $c = \max\{-a, b\}$; *or* $a = 1$, *then equality* (4) *holds.*

Remarks. In the case that $a = 0$, the left-hand side of (5) and (6) is interpreted as $\lim_{a\to 0}[{}_2F_1(-a, b; c : r)]^{1/a} = \exp(-\mu(r))$, where $r\mu'(r) = {}_2F_1(1, b; c : r) - 1$, $\mu(0) = 0$ (limit calculated using [14, (57), p. 443]). If c is strictly between $1-2a$ and $2b$ (with $1 > a$ and $c > \max\{-a, b\}$), then

$$r \mapsto \left[{}_2F_1(-a, b; c : r)\right]^{1/a} - M_\lambda\left(\frac{b}{c}, r\right)$$

is not necessarily of constant sign.

In addition to obtaining (2), Carlson [12] also observed that $a \mapsto [{}_2F_1(-a, b; c : r)]^{1/a}$ is increasing. We state the following immediate corollary which applies these noted results of Carlson in combination with the above Theorem 1.

Corollary 2 (see [12]). *Suppose* $b > 0$, $\rho \geqslant 1 \geqslant a \geqslant \sigma$, *and* $c \geqslant \max\{-a, b\}$. *If* $c \geqslant \max\{1-2a, 2b\}$ *and* $\lambda \leqslant (a+c)/(1+c)$, *then*

$$M_\rho\left(\frac{b}{c}, r\right) \geqslant \left[{}_2F_1(-\rho, b; c : r)\right]^{1/\rho} \geqslant \left[{}_2F_1(-a, b; c : r)\right]^{1/a} \geqslant M_\lambda\left(\frac{b}{c}, r\right),$$

for all $r \in (0,1)$. *If* $c \leqslant \min\{1-2a, 2b\}$ *and* $\lambda \geqslant (a+c)/(1+c)$, *then*

$$M_\sigma\left(\frac{b}{c}, r\right) \leqslant \left[{}_2F_1(-\sigma, b; c : r)\right]^{1/\sigma} \leqslant \left[{}_2F_1(-a, b; c : r)\right]^{1/a} \leqslant M_\lambda\left(\frac{b}{c}, r\right),$$

for all $r \in (0,1)$. ($\lambda = (a+c)/(1+c)$ *is sharp.*)

With the aim of verifying Theorem 1, we will first prove the following

Lemma 3. *Suppose* $\alpha > 0$, $\gamma > \max\{\alpha, 1\}$, *and* $n \in \mathbb{N}$. *Then*

$$(\gamma - 2\alpha)\left[\frac{(\alpha)_n(\frac{\gamma-1}{2})_n}{(\gamma)_n} - \left(-\frac{\alpha}{\gamma}\right)_n \cdot {}_3F_2\left(-n, \alpha, \frac{\gamma+1}{2}; \gamma, 1 + \frac{\alpha}{\gamma} - n; 1\right)\right] \geqslant 0.$$

Lemma 4. *Suppose* $\alpha, \beta > 0$, $\gamma > \max\{\alpha, 1\}$, *and* $n \in \mathbb{N}$. *If* $\gamma \geqslant \max\{2\alpha, 2\beta + 1\}$ *or* $\gamma \leqslant \min\{2\alpha, 2\beta + 1\}$, *then*

$$(\gamma - 2\alpha)\left[\frac{(\alpha)_n(\beta)_n}{(\gamma)_n} - \left(-\frac{\alpha}{\gamma}\right)_n \cdot {}_3F_2\left(-n, \alpha, \beta + 1; \gamma, 1 + \frac{\alpha}{\gamma} - n : 1\right)\right] \geqslant 0.$$

Here

$${}_3F_2(a_1, a_2, a_3; b_1, b_2 : r) \equiv \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k}{k!(b_1)_k(b_2)_k} r^k.$$

3. Proofs

Proof of Lemma 3. Suppose $\alpha > 0$, $\gamma > \max\{\alpha, 1\}$, and let $\beta = \frac{\gamma-1}{2}$, $\delta = \frac{\alpha}{\gamma}$. For $n \in \mathbb{N}$, define

$$G_n \equiv {}_3F_2(-n, \alpha - \beta - \delta, 1 + \beta; 1 + \beta - \delta, 1 - \beta - n : 1),$$
$$T_n(x) \equiv 1 - \frac{(x-\delta)_n(\gamma)_n}{(\alpha)_n(x+\beta)_n} {}_3F_2\big(-n, \alpha, \beta + 1; \gamma, 1 - (x - \delta) - n : 1\big),$$

and

$$\phi_n \equiv \frac{(1 + \beta - \delta)_n}{(\alpha)_n}.$$

The proof makes use of the following three key relationships:

$$G_{n+1} = \frac{\alpha - \delta + n}{\beta + n} G_n - \frac{\delta(1 - 2\delta)}{(\alpha + n)\phi_{n+1}}\big(1 - T_n(1)\big), \tag{7}$$
$$T_n(0) = 1 - \phi_n G_n, \tag{8}$$
$$\delta(n + \beta)\big(1 - T_n(1)\big) - (n + \delta\beta) \geqslant \beta(1 - \delta)T_n(0), \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

To verify (7), we apply the contiguous relation

$$\begin{aligned} b_2 \cdot {}_3F_2(a_1, a_2, a_3; b_1, b_2 : 1) &= a_2 \cdot {}_3F_2(a_1 + 1, a_2 + 1, a_3; b_1, b_2 + 1 : 1) \\ &\quad + \left(\frac{-a_2 a_3}{b_1}\right) \cdot {}_3F_2(a_1 + 1, a_2 + 1, a_3 + 1; b_1 + 1, b_2 + 1 : 1) \\ &\quad + (b_2 - a_2) \cdot {}_3F_2(a_1 + 1, a_2, a_3; b_1, b_2 + 1 : 1) \end{aligned}$$

(see [14, (34), p. 440]) with $a_1 = -n - 1$, $a_2 = \alpha - \beta - \delta$, $a_3 = 1 + \beta$, $b_1 = 1 + \beta - \delta$, $b_2 = 1 - (\beta + 1) - n$ to arrive at Eq. (10). The relation

$$a_3 \cdot {}_3F_2(a_1, a_2, a_3 + 1; b_1 + 1, b_2 : 1) - b_1 \cdot {}_3F_2(a_1, a_2, a_3; b_1, b_2 : 1)$$
$$= (a_3 - b_1) \cdot {}_3F_2(a_1, a_2, a_3; b_1 + 1, b_2 : 1)$$

(see [14, (26), p. 440]) is then used in Eq. (11) with $a_1 = -n$, $a_2 = 1 + \alpha - \beta - \delta$, $a_3 = 1 + \beta$, $b_1 = 1 + \beta - \delta$, $b_2 = 1 - \beta - n$. Finally the identity

$$\begin{aligned} &F(-n, a, b; c, d : 1) \\ &\quad = \frac{(c-b)_n (d-b)_n}{(c)_n (d)_n} \\ &\qquad \times {}_3F_2(-n, a + b - c - d - n + 1, b; b - c - n + 1, b - d - n + 1 : 1) \end{aligned}$$

(see [14, (81), p. 539]) is applied in Eq. (12) with $a = 1 + \alpha - \beta - \delta$, $b = 1 + \beta$, $c = 2 + \beta - \delta$, $d = 1 - \beta - n$ (using $\gamma = 2\beta + 1$). These facts yield

$$\begin{aligned} G_{n+1} &= {}_3F_2\bigl(-n-1, \alpha - \beta - \delta, 1 + \beta; 1 + \beta - \delta, 1 - (\beta + 1) - n : 1\bigr) \\ &= \frac{\delta + \beta - \alpha}{\beta + n} {}_3F_2(-n, 1 + \alpha - \beta - \delta, 1 + \beta; 1 + \beta - \delta, 1 - \beta - n : 1) \\ &\quad + \frac{(1+\beta)(\alpha - \beta - \delta)}{(1 + \beta - \delta)(\beta + n)} \\ &\quad \times {}_3F_2(-n, 1 + \alpha - \beta - \delta, 2 + \beta; 2 + \beta - \delta, 1 - \beta - n : 1) \\ &\quad + \frac{\alpha - \delta + n}{\beta + n} {}_3F_2(-n, \alpha - \beta - \delta, 1 + \beta; 1 + \beta - \delta, 1 - \beta - n : 1) \qquad (10) \\ &= \frac{\alpha - \delta + n}{\beta + n} G_n - \frac{\delta + \beta - \alpha}{(\beta + n)(1 + \beta - \delta)} \\ &\quad \times \bigl[(1 + \beta) \cdot {}_3F_2(-n, 1 + \alpha - \beta - \delta, 2 + \beta; 2 + \beta - \delta, 1 - \beta - n : 1) \\ &\quad + (-1)(1 + \beta - \delta) \\ &\quad \times {}_3F_2(-n, 1 + \alpha - \beta - \delta, 1 + \beta; 1 + \beta - \delta, 1 - \beta - n : 1)\bigr] \\ &= \frac{\alpha - \delta + n}{\beta + n} G_n - \frac{\delta(\delta + \beta - \alpha)}{(\beta + n)(1 + \beta - \delta)} \\ &\quad \times {}_3F_2(-n, 1 + \alpha - \beta - \delta, 1 + \beta; 2 + \beta - \delta, 1 - \beta - n : 1) \qquad (11) \\ &= \frac{\alpha - \delta + n}{\beta + n} G_n - \frac{\delta(\delta + \beta - \alpha)}{(\beta + n)(1 + \beta - \delta)} \frac{(1 - \delta)_n (1 - \gamma - n)_n}{(2 + \beta - \delta)_n (1 - \beta - n)_n} \\ &\quad \times {}_3F_2(-n, \alpha, 1 + \beta; \gamma, \delta - n : 1) \qquad (12) \\ &= \frac{\alpha - \delta + n}{\beta + n} G_n - \frac{\delta(\delta + \beta - \alpha)}{\beta + n} \frac{(\gamma)_n (1 - \delta)_n}{(1 + \beta - \delta)_{n+1} (\beta)_n} \\ &\quad \times {}_3F_2(-n, \alpha, 1 + \beta; \gamma, \delta - n : 1) \\ &= \frac{\alpha - \delta + n}{\beta + n} G_n - \frac{\delta(\delta + \beta - \alpha)}{\beta(\alpha + n)\phi_{n+1}} \frac{(\gamma)_n (1 - \delta)_n}{(\alpha)_n (\beta + 1)_n} \\ &\quad \times {}_3F_2\bigl(-n, \alpha, 1 + \beta; \gamma, 1 - (1 - \delta) - n : 1\bigr) \\ &= \frac{\alpha - \delta + n}{\beta + n} G_n - \frac{\delta(1 - 2\delta)}{(\alpha + n)\phi_{n+1}} \bigl(1 - T_n(1)\bigr), \quad \text{which verifies (7).} \end{aligned}$$

Recalling that $\gamma = 2\beta + 1$ and applying $(\tau)_{n-k}(1-\tau-n)_k = (-1)^k(\tau)_n$ along with the relation

$$
\begin{aligned}
&{}_3F_2(-n, a, b; c, d : 1)\\
&\quad = (-1)^n \frac{(d-a)_n(d-b)_n}{(c)_n(d)_n}\,{}_3F_2(-n, 1-d-n, a+b-c-d-n+1;\\
&\qquad\qquad\qquad\qquad\qquad a-d-n+1, b-d-n+1 : 1)
\end{aligned}
$$

(see [14, (85), p. 539]) with $a = \alpha$, $b = \beta + 1$, $c = \gamma$, $d = 1 + \delta - n$ in Eq. (13); and then the relation

$$
{}_3F_2(-n, a, b; c, d : 1) = \frac{(c-a)_n}{(c)_n} \cdot {}_3F_2(-n, a, d-b; d, 1+a-c-n : 1)
$$

(see [14, (86), p. 539]) with $a = \alpha - \beta - \delta$, $b = -\delta$, $c = \alpha - \delta$, $d = 1 + \beta - \delta$ in (14), we find that

$$
\begin{aligned}
T_n(0) &= 1 - \frac{(-\delta)_n(\gamma)_n}{(\alpha)_n(\beta)_n}\,{}_3F_2(-n, \alpha, \beta+1; \gamma, 1+\delta-n : 1)\\
&= 1 - \frac{(-1)^n(1-(1+\beta-\delta)-n)_n(1-(\alpha-\delta)-n)_n}{(\gamma)_n(1+\delta-n)_n}\,\frac{(-\delta)_n(\gamma)_n}{(\alpha)_n(\beta)_n}\\
&\quad \times {}_3F_2(-n, -\delta, \alpha-\beta-\delta; \alpha-\delta, 1+\beta-\delta : 1) \qquad (13)\\
&= 1 - \frac{(1+\beta-\delta)_n(\alpha-\delta)_n}{(\alpha)_n(\beta)_n} \cdot {}_3F_2(-n, \alpha-\beta-\delta, -\delta; \alpha-\delta, 1+\beta-\delta : 1)\\
&= 1 - \frac{(1+\beta-\delta)_n}{(\alpha)_n}\\
&\quad \times {}_3F_2(-n, \alpha-\beta-\delta, 1+\beta; 1+\beta-\delta, 1-\beta-n : 1) \qquad (14)\\
&= 1 - \phi_n G_n. \quad \text{Thus (8) holds.}
\end{aligned}
$$

Straightforward simplification using $(\tau)_{n-k}(1-\tau-n)_k = (-1)^k(\tau)_n$ yields

$$
\begin{aligned}
&\delta(n+\beta)\bigl(1 - T_n(1)\bigr) - (n+\delta\beta)\\
&\quad = \delta(n+\beta)\left[\frac{(1-\delta)_n(\gamma)_n}{(\alpha)_n(\beta+1)_n}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha)_k(\beta+1)_k(1-\delta)_{n-k}}{(\gamma)_k(1-\delta)_n}\right] - (n+\delta\beta)\\
&\quad = \delta(n+\beta)\left[1 + \frac{(\gamma)_n}{(\alpha)_n(\beta+1)_n}\sum_{k=0}^{n-1}\binom{n}{k}\frac{(\alpha)_k(\beta+1)_k(1-\delta)_{n-k}}{(\gamma)_k}\right] - (n+\delta\beta)\\
&\quad = n(\delta-1) + \frac{(\gamma+1)_{n-1}}{(\alpha+1)_{n-1}(\beta+1)_{n-1}}\sum_{k=0}^{n-1}\binom{n}{k}\frac{(\alpha)_k(\beta+1)_k(1-\delta)_{n-k}}{(\gamma)_k}\\
&\quad = (1-\delta)\left[-n + \frac{(\gamma+1)_{n-1}}{(\alpha+1)_{n-1}(\beta+1)_{n-1}}\sum_{k=0}^{n-1}\binom{n}{k}\frac{(\alpha)_k(\beta+1)_k(2-\delta)_{n-k-1}}{(\gamma)_k}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant (1-\delta)\left[-n+\frac{(\gamma+1)_{n-1}}{(\alpha+1)_{n-1}(\beta+1)_{n-1}}\sum_{k=0}^{n-1}\binom{n}{k}\frac{(\alpha)_k(\beta+1)_k(1-\delta)_{n-k-1}}{(\gamma)_k}\right]\\
&=\beta(1-\delta)\left[-\frac{n}{\beta}+\frac{(\gamma+1)_{n-1}}{(\alpha+1)_{n-1}(\beta)_n}\sum_{k=0}^{n-1}\binom{n}{k}\frac{(\alpha)_k(\beta+1)_k(1-\delta)_{n-k-1}}{(\gamma)_k}\right]\\
&=\beta(1-\delta)\left[1-\frac{n+\beta}{\beta}-\frac{(\gamma)_n}{(\alpha)_n(\beta)_n}\sum_{k=0}^{n-1}\binom{n}{k}\frac{(\alpha)_k(\beta+1)_k(-\delta)_{n-k}}{(\gamma)_k}\right]\\
&=\beta(1-\delta)\left[1-\frac{(\gamma)_n(-\delta)_n}{(\alpha)_n(\beta)_n}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha)_k(\beta+1)_k(-\delta)_{n-k}}{(\gamma)_k(-\delta)_n}\right]\\
&=\beta(1-\delta)T_n(0). \quad \text{This verifies (9).}
\end{aligned}
$$

Now to complete the proof of Lemma 3, we note that

$$
G_1=\frac{\alpha\gamma}{\gamma\beta+\gamma-\alpha}=\frac{1}{\phi_1},
$$

by direct computation. Hence $T_1(0)=1-\phi_1 G_1=0$. Now suppose $(1-2\delta)T_n(0)\geqslant 0$ for some $n\in\mathbb{N}$. Then, using (8) and (7), we have

$$
\begin{aligned}
T_{n+1}(0)-T_n(0)&=\phi_n G_n-\phi_{n+1}G_{n+1}\\
&=\phi_n G_n-\phi_{n+1}\left(\frac{\alpha-\delta+n}{\beta+n}G_n-\frac{\delta(1-2\delta)}{(\alpha+n)\phi_{n+1}}\bigl(1-T_n(1)\bigr)\right)\\
&=\frac{\delta(1-2\delta)}{\alpha+n}\bigl(1-T_n(1)\bigr)-\phi_n G_n\left(\frac{(\alpha-\delta+n)(1+\beta-\delta+n)}{(\beta+n)(\alpha+n)}-1\right)\\
&=\frac{\delta(1-2\delta)}{\alpha+n}\bigl(1-T_n(1)\bigr)-\bigl(1-T_n(0)\bigr)\frac{(1-2\delta)(n+\delta\beta)}{(\beta+n)(\alpha+n)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(1-2\delta)\bigl(T_{n+1}(0)-T_n(0)\bigr)\\
&\quad=\frac{(1-2\delta)^2}{(\alpha+n)(\beta+n)}\bigl[\delta(n+\beta)\bigl(1-T_n(1)\bigr)-(n+\delta\beta)\bigl(1-T_n(0)\bigr)\bigr]\\
&\quad=\frac{(1-2\delta)^2}{(\alpha+n)(\beta+n)}\bigl[(n+\delta\beta)T_n(0)+\delta(n+\beta)\bigl(1-T_n(1)\bigr)-(n+\delta\beta)\bigr]\\
&\quad\geqslant\frac{(1-2\delta)^2}{(\alpha+n)(\beta+n)}\bigl[(n+\delta\beta)T_n(0)+\beta(1-\delta)T_n(0)\bigr]\\
&\quad=\frac{(1-2\delta)^2}{\alpha+n}T_n(0),
\end{aligned}
$$

where the inequality follows from (9). Recalling that $\delta=\alpha/\gamma$, we now find that

$$
(1-2\delta)T_{n+1}(0)\geqslant(1-2\delta)T_n(0)\cdot\left(\frac{1-2\delta}{\alpha+n}+1\right)
$$

$$
\begin{aligned}
&= \frac{1-2\delta}{\gamma(\alpha+n)} T_n(0)\cdot\big(\gamma(\alpha+n)+(\gamma-2\alpha)\big)\\
&\geqslant \frac{1-2\delta}{\gamma(\alpha+n)} T_n(0)\cdot\big(\alpha(\alpha+n)-\alpha\big)\\
&= \frac{\alpha(1-2\delta)}{\gamma(\alpha+n)} T_n(0)\cdot(\alpha+n-1)\geqslant 0
\end{aligned}
$$

using the inductive hypothesis and the fact that $\gamma > \alpha$. Thus $(1-2\delta)T_n(0) \geqslant 0$ for all $n \in \mathbb{N}$. Hence

$$
\begin{aligned}
&(\gamma-2\alpha)\left[\frac{(\alpha)_n(\frac{\gamma-1}{2})_n}{(\gamma)_n} - (-\delta)_n \cdot {}_3F_2\left(-n, \alpha, \frac{\gamma+1}{2}; \gamma, 1+\delta-n; 1\right)\right]\\
&\quad = \frac{(\alpha)_n(\frac{\gamma-1}{2})_n}{(\gamma)_n}\gamma(1-2\delta)T_n(0) \geqslant 0, \qquad \text{for all } n \in \mathbb{N},
\end{aligned}
$$

which proves Lemma 3. □

Proof of Lemma 4. Suppose $\alpha, \beta > 0$, $\gamma > \max\{\alpha, 1\}$ with either $\gamma \geqslant \max\{2\alpha, 2\beta+1\}$ or $\gamma \leqslant \min\{2\alpha, 2\beta+1\}$. For $n \in \mathbb{N}$, define

$$
R_n(\alpha,\beta,\gamma) \equiv \beta - \frac{(-\alpha/\gamma)_n(\gamma)_n}{(\alpha)_n(\beta+1)_{n-1}} {}_3F_2(-n, \alpha, \beta+1; \gamma, 1+\alpha/\gamma-n : 1),
$$

and

$$
\begin{aligned}
S_n(\alpha,\beta,\gamma) &\equiv \frac{(\alpha)_n(\beta+1)_{n-1}}{(\gamma)_n}(\gamma-2\alpha)R_n(\alpha,\beta,\gamma)\\
&= (\gamma-2\alpha)\left[\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\right.\\
&\qquad \left. - (-\alpha/\gamma)_n \cdot {}_3F_2(-n, \alpha, \beta+1; \gamma, 1+\alpha/\gamma-n : 1)\right].
\end{aligned}
$$

Direct simplification yields

$$
\begin{aligned}
R_n(\alpha,\beta,\gamma) &= \beta - \frac{(-\alpha/\gamma)_n(\gamma)_n}{(\alpha)_n(\beta+1)_{n-1}} \sum_{k=0}^{n} \frac{(-n)_k(\alpha)_k(\beta+1)_k}{k!(\gamma)_k(1+\alpha/\gamma-n)_k}\\
&= -n + \frac{(\gamma+1)_{n-1}}{(\alpha+1)_{n-1}(\beta+1)_{n-1}} \sum_{k=0}^{n-1} \frac{n!(\alpha)_k(\beta+1)_k(1-\alpha/\gamma)_{n-k-1}}{k!(n-k)!(\gamma)_k}.
\end{aligned}
$$

For $k = 0, 1, \ldots, n-2$, $\beta \mapsto \frac{(\beta+1)_k}{(\beta+1)_{n-1}}$ is clearly a decreasing function of β. Since R_n inherits this property and either (i) $\gamma - 2\alpha \geqslant 0$ and $\frac{\gamma-1}{2} \geqslant \beta$; or (ii) $\gamma - 2\alpha \leqslant 0$ and $\frac{\gamma-1}{2} \leqslant \beta$, it follows that

$$
\begin{aligned}
S_n(\alpha,\beta,\gamma) &= \frac{(\alpha)_n(\beta+1)_{n-1}}{(\gamma)_n}(\gamma-2\alpha)R_n(\alpha,\beta,\gamma)\\
&\geqslant \frac{(\alpha)_n(\beta+1)_{n-1}}{(\gamma)_n}(\gamma-2\alpha)R_n\left(\alpha, \frac{\gamma-1}{2}, \gamma\right)
\end{aligned}
$$

$$= \frac{(\beta+1)_{n-1}}{(\frac{\gamma-1}{2})_n} S_n\left(\alpha, \frac{\gamma-1}{2}, \gamma\right) \geqslant 0,$$

for all $n \in \mathbb{N}$, by Lemma 3. This proves Lemma 4. □

Proof of Theorem 1. Suppose $b > 0$, $1 > a \neq 0$, $c > \max\{-a, b\}$ with either $c \geqslant \max\{1-2a, 2b\}$ or $c \leqslant \min\{1-2a, 2b\}$. Let $\lambda = \frac{a+c}{1+c}$ and define $\sum_{n=0}^{\infty} A_n r^n \equiv {}_2F_1(-a, b; c; r)$ and $H(r) \equiv \sum_{n=0}^{\infty} B_n r^n \equiv [M_\lambda(\frac{b}{c}, r)]^a$. Paralleling and then extending an inductive argument used in [6], we will show $a(2\lambda-1)B_n \leqslant a(2\lambda-1)A_n$ for all $n \in \mathbb{N}$. It easily follows that $B_0 = A_0$ and $B_1 = A_1 = \frac{-ab}{c}$. Now suppose that $a(2\lambda-1)B_k \leqslant a(2\lambda-1)A_k$ for all $k = 1, \ldots, n$. The logarithmic derivative of H becomes

$$\frac{H'(r)}{H(r)} = \frac{-ab}{(c-b)(1-r)^{1-\lambda} + b(1-r)},$$

and thus

$$\left[(c-b)(1-r)^{1-\lambda} + b(1-r)\right] \sum_{n=0}^{\infty} (n+1)B_{n+1} r^n = -ab \sum_{n=0}^{\infty} B_n r^n.$$

Applying the Cauchy product to the previous equation with $(1-r)^\delta = \sum_{n=0}^{\infty} \frac{(-\delta)_n}{n!} r^n$, we find that

$$\begin{aligned}
&ac(2\lambda-1)(n+1)B_{n+1}\\
&\quad = a(2\lambda-1)\left[bB_n(n-a) - (c-b)\sum_{k=0}^{n-1}(k+1)B_{k+1}\frac{(\lambda-1)_{n-k}}{(1)_{n-k}}\right]\\
&\quad = a(2\lambda-1)\left[bB_n(n-a) + (1-\lambda)(c-b)\sum_{k=0}^{n-1}(k+1)B_{k+1}\frac{(\lambda)_{n-k-1}}{(1)_{n-k}}\right]\\
&\quad \leqslant a(2\lambda-1)\left[bA_n(n-a) + (1-\lambda)(c-b)\sum_{k=0}^{n-1}(k+1)A_{k+1}\frac{(\lambda)_{n-k-1}}{(1)_{n-k}}\right]\\
&\quad = a(2\lambda-1)\left[bA_n(n-a) - (c-b)\sum_{k=0}^{n}(k+1)A_{k+1}\frac{(\lambda-1)_{n-k}}{(1)_{n-k}}\right]\\
&\qquad + a(2\lambda-1)(c-b)(n+1)A_{n+1}\\
&\quad = ac(2\lambda-1)(n+1)A_{n+1} + ab(2\lambda-1)\left[A_n(n-a) - (n+1)A_{n+1}\right]\\
&\qquad + a(2\lambda-1)(b-c)\sum_{k=0}^{n}(k+1)\frac{(-a)_{k+1}(b)_{k+1}(\lambda-1)_{n-k}}{(k+1)!(c)_{k+1}(1)_{n-k}}\\
&\quad = ac(2\lambda-1)(n+1)A_{n+1} + ab(2\lambda-1)\left[A_n(n-a) - (n+1)A_{n+1}\right]\\
&\qquad + \frac{a^2b(2\lambda-1)(c-b)}{c}\sum_{k=0}^{n}\frac{(1-a)_k(b+1)_k(\lambda-1)_{n-k}}{k!(c+1)_k(1)_{n-k}}
\end{aligned}$$

since $A_n = \frac{(-a)_n(b)_n}{n!(c)_n}$. Again using $(\tau)_{n-k}(1-\tau-n)_k = (-1)^k(\tau)_n$, we note that

$$\sum_{k=0}^{n}\frac{(1-a)_k(b+1)_k(\lambda-1)_{n-k}}{k!(c+1)_k(1)_{n-k}}=\frac{(\lambda-1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(1-a)_k(b+1)_k}{k!(c+1)_k(2-\lambda-n)_n}$$
$$=\frac{(\lambda-1)_n}{n!}{}_3F_2(-n,1-a,b+1;c+1,2-\lambda-n;1),$$

and

$$ab(2\lambda-1)\big[A_n(n-a)-(n+1)A_{n+1}\big]=-\frac{a^2b(c-b)(1-a)_n(b)_n(2\lambda-1)}{c(c+1)_n n!}.$$

Therefore

$$\begin{aligned}
&ac(2\lambda-1)(n+1)B_{n+1}\\
&\quad\leqslant ac(2\lambda-1)(n+1)A_{n+1}-\frac{a^2b(c-b)(1-a)_n(b)_n(2\lambda-1)}{c(c+1)_n n!}\\
&\qquad+\frac{a^2b(c-b)(2\lambda-1)}{cn!}(\lambda-1)_n\cdot{}_3F_2(-n,1-a,b+1;c+1,2-\lambda-n;1)\\
&\quad=ac(2\lambda-1)(n+1)A_{n+1}+\frac{a^2b(c-b)}{cn!}(2\lambda-1)\\
&\qquad\times\left[(\lambda-1)_n\cdot{}_3F_2(-n,1-a,b+1;c+1,2-\lambda-n;1)-\frac{(1-a)_n(b)_n}{(c+1)_n}\right]\\
&\quad=ac(2\lambda-1)(n+1)A_{n+1}+\frac{a^2b(c-b)}{cn!}(1-2\delta)\\
&\qquad\times\left[(-\delta)_n\cdot{}_3F_2(-n,\alpha,\beta+1;\gamma,1+\delta-n;1)-\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\right]\\
&\quad\leqslant ac(2\lambda-1)(n+1)A_{n+1}
\end{aligned}$$

using Lemma 4 with $\alpha=1-a$, $\beta=b$, $\gamma=c+1$, and $\delta=1-\lambda=\frac{\alpha}{\gamma}$. Thus $a(2\lambda-1)B_n\leqslant a(2\lambda-1)A_n$ for all $n\in\mathbb{N}$, by induction. Using this result together with $(2\lambda-1)=(2a+c-1)/(c+1)$, we find that

$$a\left[M_\lambda\left(\frac{b}{c},r\right)\right]^a\leqslant a\cdot{}_2F_1(-a,b;c:r),\quad\text{when } c\geqslant\max\{1-2a,2b\},\tag{15}$$
$$a\left[M_\lambda\left(\frac{b}{c},r\right)\right]^a\geqslant a\cdot{}_2F_1(-a,b;c:r),\quad\text{when } c\leqslant\min\{1-2a,2b\}.\tag{16}$$

After multiplying both sides of (15) and (16) by $1/a$ and then taking the ath root, one finds that this verifies (5) and (6) in the special case that $\lambda=\frac{a+c}{1+c}$. The monotonicity of $\lambda\mapsto M_\lambda$ now implies that (5) holds for all $\lambda\leqslant\frac{a+c}{1+c}$ and (6) holds for all $\lambda\geqslant\frac{a+c}{1+c}$. Sharpness follows from the observation that

$$a\left({}_2F_1(-a,b;c:r)-\left[M_\lambda\left(\frac{b}{c},r\right)\right]^a\right)$$
$$=\frac{a^2b(c-b)}{2}\left[\frac{a+c-\lambda(1+c)}{c^2(c+1)}\right]r^2+O(r^3).$$

Thus $\lambda\leqslant\frac{a+c}{1+c}$ is a necessary condition for (5), and $\lambda\geqslant\frac{a+c}{1+c}$ is necessary for (6). □

4. Concluding remarks

It is interesting to note that H. Alzer and S.-L. Qiu [3] have verified Alzer's conjecture that the inequality in (1) involving the complete elliptic integral reverses if and only if $\lambda \geqslant \ln(\sqrt{2})/\ln(\pi/2) \approx 0.7675$. Generalizations of this and other sharp inequalities complementary to Theorem 1 remain as intriguing problems for further study.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.
[2] G. Almkvist, B. Berndt, Gauss, Landen, Ramanujan, the arithmetic–geometric mean, ellipses, π, and the Ladies Diary, Amer. Math. Monthly 95 (1988) 585–608.
[3] H. Alzer, S.-L. Qiu, Monotonicity theorems and inequalities for the complete elliptic integrals, J. Comput. Appl. Math. 172 (2004) 289–312.
[4] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Conformal Invariants, Inequalities, and Quasiconformal Maps, Wiley, New York, 1997.
[5] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge Univ. Press, Cambridge, 1999.
[6] R.W. Barnard, K.C. Richards, A note on the hypergeometric mean value, Comput. Methods Function Theory 1 (2001) 81–88.
[7] R.W. Barnard, K. Pearce, K.C. Richards, A monotonicity property involving ${}_3F_2$ and comparisons of classical approximations of elliptical arc length, SIAM J. Math. Anal. 32 (2000) 403–419.
[8] R.W. Barnard, K. Pearce, K.C. Richards, An inequality involving the generalized hypergeometric function and the arc length of an ellipse, SIAM J. Math. Anal. 31 (2000) 693–699.
[9] E. Beckenbach, R. Bellman, Inequalities, fourth ed., Springer-Verlag, Berlin, 1983.
[10] J. Borwein, P. Borwein, Inequalities for compound mean iterations with logarithmic asymptotes, J. Math. Anal. Appl. 177 (1993) 572–582.
[11] P.S. Bullen, D.S. Mitrinović, P.M. Vasć, Means and Their Inequalities, Reidel, Dordrecht, 1988.
[12] B.C. Carlson, Some inequalities for hypergeometric functions, Proc. Amer. Math. Soc. 16 (1966) 32–39.
[13] I.S. Gradshteyn, I.M. Ryzhik, in: A. Jeffrey (Ed.), Tables of Integrals and Series, fifth ed., Academic Press, San Diego, 1994.
[14] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. 3, Gordon and Breach, New York, 1990.
[15] M. Vuorinen, Hypergeometric functions in geometric function theory, in: K.R. Srinivasa, R. Jagannthan, G. Van der Jeugy (Eds.), Proc. Special Functions and Differential Equations, Allied Publishers, New Delhi, 1998.