Noname manuscript No.
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Complete r-partite graphs determined by their domination polynomial

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the date of receipt and acceptance should be inserted later

Abstract The domination polynomial of a graph is the polynomial whose coefficients count the number of dominating sets of each cardinality. A recent question asks which graphs are uniquely determined (up to isomorphism) by their domination polynomial. In this paper, we completely describe the complete r-partite graphs which are; in the bipartite case, this settles in the affirmative a conjecture of Aalipour, Akbari and Ebrahimi [2].

Keywords domination polynomial · dominating set · $\mathcal{D}$-unique graphs

Mathematics Subject Classification (2000) 05C69 · 05C75

1 Introduction

While some graph-invariant polynomials are well-studied (including the chromatic polynomial for a century [9], the independence polynomial for decades [11]), other graph-invariant problems have not benefited from years of exploration. Recently there has been growing interest in the domination polynomial of a graph (see, for instance [2], [3], [7], [8]). We consider in particular a research problem posed at the 22nd British Combinatorial Conference (problem 519 in [10]) in July 2009. We solve that problem, proving the conjecture of Aalipour, Akbari and Ebrahimi [2] in the affirmative, and provide some additional results about domination polynomials.

We begin with some notation. For a simple graph $G = (V, E)$, we let $v(G) = |V|$ and $e(G) = |E|$ denote the order and size, respectively, of G. For

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each vertex $v \in V$, we let $N_G(v)$ and $d_G(v) = |N_G(v)|$ denote the neighborhood and degree, respectively, of v in G, and $N_G[v] = N_G(v) \cup \{v\}$ denote the closed neighborhood of v in G. We drop the subscript when the graph G is clear from the context.

A set of vertices $S \subseteq V$ is said to be a *dominating set*, or that S *dominates* G, if $\bigcup_{x \in S} N[x] = V$, i.e., every vertex in $V \setminus S$ has a neighbor in S. Similarly, we call a set of vertices $T \subseteq V$ which does not dominate a *non-dominating set*, or say that T does not dominate G. It is immediate that a subset of a non-dominating set is also a non-dominating set.

Let $d(G, i)$ denote the number of dominating sets in G of cardinality i. Note that for non-empty G, $d(G, 0) = 0$. The *domination polynomial* of G, introduced in [4,7], is the polynomial

$$D(G, x) = \sum_{i=1}^{v(G)} d(G, i) \cdot x^i \quad .$$

This polynomial is easily seen to be a graph invariant and subsequently induces a natural equivalence relation on graphs. Two graphs G and H are deemed to be $\mathcal{D}$-*equivalent* if $D(G, x) = D(H, x)$. Letting $[G]$ denote the equivalence class of G up to isomorphism under this relation, we say G is $\mathcal{D}$-*unique* if $[G] = \{G\}$, i.e., if $D(H, x) = D(G, x)$ implies $H \cong G$.

A question of recent interest concerning this equivalence relation $[\cdot]$ asks which graphs are determined by their domination polynomial. It is known that cycles [3],[5] and cubic graphs of order 10 [6] (particularly, the Petersen graph) are, while if $n \equiv 0 \pmod 3$, the path of order n is not [3]. However, the question remains open for most graphs.

Aalipour-Hafshejani, Akbari and Ebrahimi [2] considered this question for the complete bipartite graph $K_{a,b}$. To state their results, we introduce some further notation. We let $\mathbb{N}$ denote the positive integers. Let $\chi(G)$ be the chromatic number of a graph G. For $t \in \{0, 1\}$ and $a \in \mathbb{N}$, let $H_t(a)$ denote the graph on $2a + t$ vertices formed by a copy of K_a and a copy of K_{a+t} connected by a matching of size a from K_a into K_{a+t}. An example is given in Figure 1. Equivalently, $H_0(a)$ is the Cartesian graph product $K_a \times K_2$, while $H_1(a)$ is the graph formed by deleting a vertex from $H_0(a + 1)$. We point out that $H_0(1) \cong K_{1,1}$, $H_0(2) \cong K_{2,2}$, and $H_1(1) \cong K_{1,2}$, but, in general, $\chi(H_t(a)) = a + t$ and hence $H_t(a) \ncong K_{a,a+t}$ for all other a, t.

Aalipour-Hafshejani *et al.* observed the following fact and showed the following result.

Fact 1 ([2]) *For any $a \geq 1$ and $t \in \{0, 1\}$, $D(K_{a,a+t}, x) = D(H_t(a), x)$.*

Theorem 1 ([2]) *For all $a \in \mathbb{N}$, $[K_{a,a}] = \{K_{a,a}, H_0(a)\}$.*

They also conjectured that $[K_{a,a+1}] = \{K_{a,a+1}, H_1(a)\}$, and established partial results supporting this conjecture. However, the absence of a similar construction when the size of the partitions differs by more than one led them to the following conjecture.

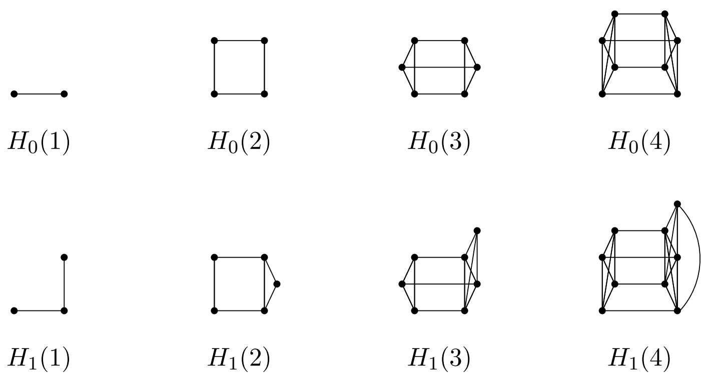

Fig. 1 The graphs $H_0(a)$ and $H_1(a)$ for $1 \leq a \leq 4$.

Conjecture 1 ([2], Conjecture 2) For all $a, b \in \mathbb{N}$, if $|a-b| \geq 2$ then $K_{a,b}$ is $\mathcal{D}$-unique.

Some partial progress on Conjecture 1 was made in [1], where it was verified in the case $a \leq 4$ or $b > \max\{\binom{a}{2}, a+2\}$. Our main result resolves Conjecture 1 in the affirmative, and, moreover, provides a complete classification of the $\mathcal{D}$-unique complete r-partite graphs. So that our notation remains consistent through the remainder of the paper, for a given $r \in \mathbb{N}$ and integers $a_1, \ldots, a_r \in \mathbb{N}$, we let $\mathcal{K}(a_1, \ldots, a_r)$ denote the complete r-partite graph on $n = a_1 + a_2 + \cdots + a_r$ vertices with color classes of sizes $a_1, \ldots, a_r$. (Note that $\mathcal{K}(a)$ denotes an independent set on a vertices rather than the complete graph K_a.)

Theorem 2 *Let $r \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_r \in \mathbb{N}$. Then the complete r-partite graph $\mathcal{K}(a_1, \ldots, a_r)$ is $\mathcal{D}$-unique if and only if for all $1 \leq i < j \leq r$, either $\max\{a_i, a_j\} \leq 2$ or $|a_i - a_j| \geq 2$.*

The proof of Theorem 2 is given in Section 3.

2 Preliminaries

We provide some additional notation that is useful in considering the equipartite case. When $a_1 = a_2 = \cdots = a_r =: a$, we simply write $\mathcal{K}_r(a)$ in place of $\mathcal{K}(a_1, \ldots, a_r)$. Thus, for example, the complete graph K_r on r vertices is $K_r = \mathcal{K}_r(1)$. We also let $\mathcal{E}$ denote the empty graph (with no vertices or edges), and we define $\mathcal{K}_0(a) = \mathcal{E}$ for all $a \in \mathbb{N}$.

We obtain a partial extension of Theorem 1 by determining the $\mathcal{D}$-equivalence class of the complete equipartite graphs $\mathcal{K}_r(a)$ for large a. To do so, we consider the *join* of graphs G and H, denoted $G \vee H$, which we emphasize is defined only when $V(G) \cap V(H) = \emptyset$. In particular, for disjoint G and H, $G \vee H$ is defined to be the graph formed by the union of G and H together with the edges in the complete bipartite graph with partitions $V(G)$ and $V(H)$.

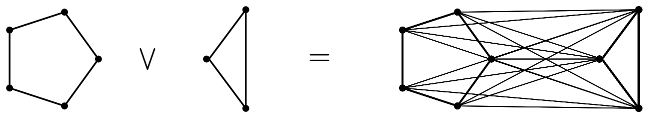

Fig. 2 The join of C_5 and K_3.

See the example in Figure 2. (Note that $G \vee \mathcal{E} = G$ for all graphs G.) We extend this definition in the obvious way to the *join* of k disjoint graphs $G_1, \ldots, G_k$, denoted by $G_1 \vee G_2 \vee \cdots \vee G_k = \bigvee_{i=1}^{k} G_i$. In particular, note that $\mathcal{K}(a_1, \ldots, a_r) = \bigvee_{i=1}^{r} \mathcal{K}(a_i)$.

Now, for $r \geq 2$ and $0 \leq t \leq \lfloor r/2 \rfloor$, we define the graph $J_r(a,t)$ on $a \cdot r$ vertices as follows. Given (vertex-disjoint) copies $H_1, \ldots, H_t$ of $H_0(a)$, let

$$J_r(a,t) = \left(\bigvee_{i=1}^{t} H_i \right) \vee \mathcal{K}_{r-2t}(a),$$

noting $J_r(a,0) = \mathcal{K}_r(a)$. Examples of these graphs when $r = 4$, $a = 3$ and $0 \leq t \leq 2$ are given in Figure 3.

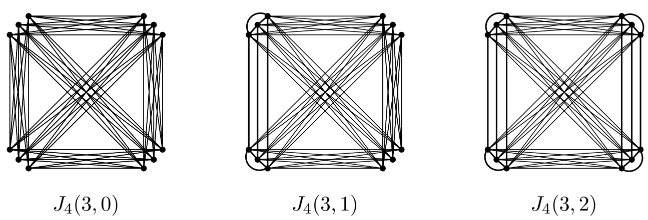

Fig. 3 The graphs $J_4(3,0)$, $J_4(3,1)$, and $J_4(3,2)$.

Theorem 3 *For $r \geq 2$ and $a \geq r+2$, $[\mathcal{K}_r(a)] = \{J_r(a,t) : 0 \leq t \leq \lfloor r/2 \rfloor\}$.*

The proof of Theorem 3 will follow in Section 4.

We remark that in this range of a, we will show that each of the graphs $J_r(a,i)$ are pairwise non-isomorphic as i varies, and, in particular, the $\mathcal{D}$-equivalence class of $\mathcal{K}_r(a)$ contains $\lfloor r/2 \rfloor + 1$ distinct graphs. We mention that our arguments also show that if r is even, Theorem 3 holds for $a \geq r+1$. In particular, Theorems 2 and 3 imply Theorem 1.

For a set X we let 2^X denote the power set of X, and for integers $k \geq 0$ we let $\binom{X}{k}$ denote the family of k-element subsets of X. For a graph $G = (V, E)$, let $\mathcal{ND}(G) \subseteq 2^V$ denote the family of non-dominating sets of vertices of G. Our arguments will rely in part on the following simple result.

Lemma 1 *For graphs G and H, $\mathcal{ND}(G \vee H) = \mathcal{ND}(G) \cup \mathcal{ND}(H)$.*

Proof Since for each $v \in V(G)$, $N_{G\vee H}(v) \supseteq V(H)$, and since for each $w \in V(H)$, $N_{G\vee H}(w) \supseteq V(G)$, a set $X \subseteq V(G) \cup V(H)$ does not dominate $G \vee H$ if and only if either $X \subseteq V(G)$ and X does not dominate G, or $X \subseteq V(H)$ and X does not dominate H. □

Since $\mathcal{K}(a)$ has no proper dominating sets, the following corollary is immediate.

Corollary 1 *$\mathcal{ND}(\mathcal{K}(a_1, \ldots, a_r))$ is the set of proper subsets of the color classes of $\mathcal{K}(a_1, \ldots, a_r)$.*

Since the family of dominating sets of G is $2^{V(G)} \setminus \mathcal{ND}(G)$, Lemma 1 also easily implies the following result from [3].

Lemma 2 ([3], Theorem 2) *For graphs G and H,*

$$D(G \vee H, x) = \left((1+x)^{v(G)} - 1\right)\left((1+x)^{v(H)} - 1\right) + D(G, x) + D(H, x).$$

Note that $((1+x)^{v(G)} - 1) \cdot ((1+x)^{v(H)} - 1)$ is the generating function for the number of subsets $X \subseteq V(G) \cup V(H)$ such that $X \cap V(G) \neq \emptyset$ and $X \cap V(H) \neq \emptyset$. Our arguments will only require the following corollary of Lemma 2.

Corollary 2 *For graphs G_1, G_2, and H, $D(G_1 \vee H, x) = D(G_2 \vee H, x)$ if and only if $D(G_1, x) = D(G_2, x)$.*

Next, we will appeal to the following lemma from [3]. Our statement differs slightly from that in [3], but the proof is identical; we include it for completeness. Informally, this result shows that the domination polynomial both determines the minimum degree $\delta(G)$ of the graph and bounds below the number of vertices of minimum degree.

Lemma 3 ([3], Lemma 4) *Let $G = (V, E)$ be a graph of order n with domination polynomial $D(G, x) = \sum_{i=1}^{n} d(G, i)x^i$, and let $\ell = \min\{j : d(G, j) = \binom{n}{j}\}$. Then $\delta(G) = n - \ell$, and for each non-dominating set T of cardinality $\ell - 1$, there exists a vertex v_T such that $N[v_T] = V \setminus T$.*

Proof Suppose first that some vertex $v \in V$ has $d(v) < n - \ell$: then $V \setminus N[v]$ has cardinality at least ℓ and does not dominate, contradicting the choice of ℓ.

Next, let $T \subseteq V$ be a non-dominating $(\ell - 1)$-set: by definition, there exists a vertex $v_T \in V$ such that $N[v_T] \cap T = \emptyset$. Since $d(v_T) \leq n - (|T| + 1) = n - \ell = \delta(G)$, equality holds and so $N[v_T] = V \setminus T$. □

Corollary 3 ([3], Theorem 14) *Let G be a graph, and suppose H satisfies $D(H, x) = D(G, x)$. Then $\delta(H) = \delta(G) = n - \ell$ for some $\ell \geq 1$, and H contains at least $\binom{n}{\ell-1} - d(G, \ell - 1)$ vertices of degree $\delta(H)$.*

Finally, we will appeal to two classical results from extremal combinatorics. The first is a special case of Turán's Theorem.

Theorem 4 (Turán, [14]) *For $r \geq 3$ and $a \in \mathbb{N}$, let G be a graph on $n = a \cdot r$ vertices and suppose G is K_{r+1}-free. Then $e(G) \leq e(\mathcal{K}_r(a))$, with equality if and only if $G \cong \mathcal{K}_r(a)$.*

The second is a variant of the Kruskal-Katona Theorem due to Lovász ([13], Problem 13.31; see also [12], Theorem 1 for a short proof). We recall that for a family $\mathcal{F} \subseteq \binom{V}{k}$ of k-element subsets of a set V, the *shadow* of $\mathcal{F}$ is the family

$$\partial\mathcal{F} = \{Y \subseteq V : |Y| = k-1, Y \subseteq X \text{ for some } X \in \mathcal{F}\} = \bigcup_{X \in \mathcal{F}} \binom{X}{k-1}.$$

Theorem 5 ([13]) *Let $k \in \mathbb{N}$ be an integer, V be a finite set, and let $\mathcal{F} \subseteq \binom{V}{k}$. If $|\mathcal{F}| = \binom{x}{k}$ for some real $x \geq k$, then*

$$|\partial\mathcal{F}| \geq \binom{x}{k-1}.$$

Moreover, if equality holds then $x \in \mathbb{N}$ and $\mathcal{F} = \binom{X}{k}$ for some $X \subseteq V$, $|X| = x$.

3 Proof of Theorem 2

We now return to our classification of the $\mathcal{D}$-unique complete r-partite graphs. Let $r \in \mathbb{N}$ and $a_1, \ldots, a_r \in \mathbb{N}$ be given. We will show that $G = \mathcal{K}(a_1, \ldots, a_r)$ is $\mathcal{D}$-unique if and only if

$$\max\{a_i, a_j\} \leq 2 \quad \text{or} \quad |a_i - a_j| \geq 2\,, \quad \forall\ 1 \leq i < j \leq r. \tag{1}$$

We start with the simple proof of the necessity, expanding the observations of Aalipour, Akbari and Ebrahimi mentioned in the introduction. Suppose that (1) fails for some $i < j$, noting this requires that $r \geq 2$. Relabelling the indices if necessary, we may assume $i = 1$, $j = 2$, and that $a_1 \leq a_2$. Since both of the conditions of the disjunction (1) fail, we must have $a_2 = \max\{a_1, a_2\} \geq 3$ and $|a_2 - a_1| = a_2 - a_1 \leq 1$. We can thus represent the difference of a_2 and a_1 as $k := a_2 - a_1 \in \{0, 1\}$. Taking $\mathcal{K}(a_3, \ldots, a_r) = \mathcal{E}$ if $r = 2$, let

$$H = H_k(a_1) \vee \mathcal{K}(a_3, \ldots, a_r),$$

and observe that $\mathcal{K}(a_1, \ldots, a_r) = \mathcal{K}(a_1, a_2) \vee \mathcal{K}(a_3, \ldots, a_r)$. Since $a_2 = a_1 + k$, Fact 1 gives that $D(H_k(a_1), x) = D(\mathcal{K}(a_1, a_2), x)$, and thus Corollary 2 yields $D(H, x) = D(\mathcal{K}(a_1, \ldots, a_r), x)$. It remains only to show that

$$H \ncong \mathcal{K}(a_1, \ldots, a_r).$$

But this, in turn, follows as $\chi(H_k(a_1)) = a_1 + k = a_2 \geq 3$, and since the chromatic number of the join of two graphs is the sum of their chromatic numbers,

$$\chi(H) = \chi(H_k(a_1)) + \chi(\mathcal{K}(a_3, \ldots, a_r)) \geq 3 + (r-2) > r = \chi(G),$$

and thus $H \ncong G$.

To show (1) suffices to ensure $\mathcal{D}$-uniqueness, we proceed by induction on r. The base case $r = 1$ is the assertion that $\mathcal{K}(a)$ is $\mathcal{D}$-unique for all $a \in \mathbb{N}$. Since $\mathcal{K}(a)$ is an independent set on a vertices, the only dominating set is its entire vertex set. From the simple observation that any graph containing an edge also contains a proper dominating set, we see that $\mathcal{K}(a)$ is in fact $\mathcal{D}$-unique for all $a \in \mathbb{N}$.

We therefore continue our inductive proof by assuming $r \geq 2$ and (1) holds. To simplify our further notation, for $G = \mathcal{K}(a_1, \ldots, a_r)$ we relabel the indices if necessary so that

$$a_1 \leq a_2 \leq \cdots \leq a_r.$$

Let $n = a_1 + a_2 + \cdots + a_r = v(G)$, and let $H = (V, E)$ be any graph satisfying $D(H, x) = D(G, x)$, which immediately implies that $v(H) = n$. Our argument proceeds by considering separately the cases $a_r \leq 2$ and $a_r > 2$.

Case 1. $a_r \leq 2$**.** Since $a_r = \max_i a_i$, the color classes of G must all be of size 1 or size 2. Letting $d = |\{i : a_i = 2\}|$ and $s = r - d$, G has s singleton color classes and d doubleton color classes. In particular, it follows from Corollary 1 that $d(G, 2) = \binom{n}{2}$ and $d(G, 1) = n - 2d$. Since $D(H, x) = D(G, x)$, it follows that H has exactly $n - 2d$ vertices of degree $n - 1$ (which dominate). Furthermore, Lemma 3 implies that $\delta(H) = n - 2$ if $d > 0$, and thus H contains exactly $2d$ vertices of degree $n - 2$. To see that this implies $H \cong G$, simply observe that the complement of each consists of $n - 2d$ isolated vertices and a matching on $2d$ vertices.

Case 2. $a_r > 2$**.** In this case, (1) implies $a_i \leq a_r - 2$ for all $i < r$. Since the only color class of G containing proper subsets of cardinality at least $a_r - 2$ is the unique color class of size a_r, Corollary 1 yields

$$d(H, a_r) = d(G, a_r) = \tbinom{n}{a_r}, \tag{2}$$

$$d(H, a_r - 1) = d(G, a_r - 1) = \tbinom{n}{a_r - 1} - \tbinom{a_r}{a_r - 1} \quad \text{and} \tag{3}$$

$$d(H, a_r - 2) = d(G, a_r - 2) = \tbinom{n}{a_r - 2} - \tbinom{a_r}{a_r - 2}. \tag{4}$$

To use these bounds, we consider the following families of non-dominating sets of H.

$$\mathcal{F} = \{X \in \mathcal{ND}(H) : |X| = a_r - 1\}, \quad \text{and} \quad \mathcal{G} = \{Y \in \mathcal{ND}(H) : |Y| = a_r - 2\}.$$

Since $X \in \mathcal{ND}(H)$ implies $2^X \subseteq \mathcal{ND}(H)$, it easily follows that $\partial\mathcal{F} \subseteq \mathcal{G}$ so by (3), (4) and Theorem 5,

$$\binom{a_r}{a_r - 2} = |\mathcal{G}| \geq |\partial\mathcal{F}| \geq \binom{a_r}{(a_r-1)-1} = \binom{a_r}{a_r-2}.$$

As equality holds throughout, it follows (again by Theorem 5) that $\mathcal{F} = \binom{A}{a_r-1}$ for some $A \subseteq V$ with $|A| = a_r$.

Next, by (2) and (3) we may apply Lemma 3 with $\ell = a_r$ to conclude that for each non-dominating set $X \in \mathcal{F}$, we may select a vertex v_X such that $N_H[v_X] = V \setminus X$. Let $Z = \{v_X : X \in \mathcal{F}\}$ be the set of these vertices, observing that

$$|Z| = |\mathcal{F}| = \binom{a_r}{a_r-1} = a_r = |A|.$$

Since $\mathcal{F} = \binom{A}{a_r-1}$, for each $v = v_X \in Z$ we have $|N[v_X] \cap A| = |A \setminus X| = 1$. This implies that each $T \subseteq Z$ with $|T| = a_r - 1$ must be a non-dominating set in H, and thus $\binom{Z}{a_r-1} \subseteq \mathcal{F} = \binom{A}{a_r-1}$, which, as $|Z| = |A|$, yields that $Z = A$.

Consequently, for each $v \in A = Z$ we have $v = v_{A\setminus\{v\}}$ and so $N_H(z) = V \setminus A$. But this implies that A is an independent set and

$$H = H[V \setminus A] \vee H[A] \cong H[V \setminus A] \vee \mathcal{K}(a_r).$$

Since $G = \mathcal{K}(a_1, \ldots, a_{r-1}) \vee \mathcal{K}(a_r)$, Corollary 2 implies $D(H[V \setminus A], x) = D(\mathcal{K}(a_1, \ldots, a_{r-1}), x)$, and then induction implies $H[V \setminus A] \cong \mathcal{K}(a_1, \ldots, a_{r-1})$, and so

$$H \cong \mathcal{K}(a_1, \ldots, a_{r-1}) \vee \mathcal{K}(a_r) \cong G,$$

completing the proof of Theorem 2. □

4 Proof of Theorem 3

We begin by fixing $r \geq 2$ and $a \geq r+2$. Let $G = \mathcal{K}_r(a)$, and let $H = (V, E)$ be any graph satisfying $D(H, x) = D(G, x)$. To prove Theorem 3 we must show that for some $0 \leq t \leq \lfloor r/2 \rfloor$ we have $H \cong J_r(a, t)$, where

$$J_r(a,t) = \left(\bigvee_{i=1}^{t} H_0(a) \right) \vee \mathcal{K}_{r-2t}(a).$$

We note that $D(J_r(a,t), x) = D(G, x)$ follows from Corollary 2 by induction, since $D(H_0(a), x) = D(\mathcal{K}(a, a), x)$. We also mention that $J_r(a,t) \ncong J_r(a,t')$ for $t \neq t'$, which can be verified by observing that the complement of $H_0(a)$ is connected for $a \geq 3$ and thus the complement of $J_r(a,t)$ consists of $r - 2t$ components of size a and t of size $2a$ each.

Let $n = a \cdot r$, so $n = v(G) = v(H)$. Since $\mathcal{K}_r(a)$ is $(n-a)$-regular and (by Corollary 1) has $r \cdot \binom{a}{a-1} = r \cdot a = n$ non-dominating sets of cardinality $a-1$, Corollary 3 implies H is $(n-a)$-regular as well. For each vertex $v \in V$, we let

$$Y_v = V \setminus N_H[v]$$

so $Y_v \in \mathcal{ND}(H)$ and $|Y_v| = a - 1$. We note that if $v \neq w$ then $Y_v \neq Y_w$, as there are n non-dominating sets of cardinality $a-1$ and as each has the form Y_v for some $v \in V$ by Lemma 3.

Our next step is to identify 'where' in the graph H the sets Y_v occur, in a sense to be made clear below. To this end, we define an auxiliary graph F on V with edge set $\{uv : uv \text{ dominates } H\}$. Since $a > 2$, the only pairs of vertices in G which dominate are its edges, implying $e(F) = d(H,2) = d(G,2) = e(\mathcal{K}_r(a))$.

Claim 1 *F is K_{r+1}-free.*

Proof (Proof of Claim 1) Suppose to the contrary that F contains a copy of K_{r+1} with vertex set $\{v_1, \ldots, v_{r+1}\}$. Since for $1 \leq i < j \leq r+1$, the pair v_iv_j dominates H if and only if $Y_{v_i} \cap Y_{v_j} = \emptyset$, we conclude

$$a \cdot r = |V| \geq \sum_{i=1}^{r+1} |Y_{v_i}| = (r+1)(a-1) = ar + [a - (r+1)],$$

a contradiction as $a \geq r + 2$. □

Thus, by Theorem 4 we have $F \cong \mathcal{K}_r(a)$, so we partition $V = V_1 \cup \cdots \cup V_r$ into the r color classes of the auxiliary graph F. Since any subset of V intersecting more than one set V_i contains an edge of F and hence dominates H, it follows that for each $v \in V$ we must have $Y_v \subseteq V_i$ for some i.

Now, for $1 \leq i \leq r$, let $U_i = \bigcup_{v \in V_i} Y_v$. Since $v \neq w$ implies $Y_v \neq Y_w$ and since $a \geq 2$, it follows that $|U_i| \geq a$. The proof of Theorem 3 is then an immediate consequence of the following claim.

Claim 2 *For each $1 \leq i \leq r$ there is a unique j (possibly with $j = i$) such that $U_i = V_j$ and $U_j = V_i$, and*

$$H[V_i \cup V_j] \cong \begin{cases} \mathcal{K}(a) & \text{if } i = j, \\ H_0(a) & \text{if } i \neq j. \end{cases}$$

Proof (Proof of Claim 2) Let $1 \leq i \leq r$ be given and let $v, w \in V_i$ be distinct vertices. Let j, k be given so that $Y_v \subseteq V_j$ and $Y_w \subseteq V_k$. Since vw is not an edge of F and hence does not dominate H, $Y_v \cap Y_w \neq \emptyset$ follows, implying $k = j$ and, letting $w \in V_i \setminus \{v_j\}$ vary, that $U_i \subseteq V_j$. Since $|V_j| = a \leq |U_i|$, $U_i = V_j$ follows.

Now, if $i = j$, then for each $v \in V_i$ we have $V_i = \{v\} \cup Y_v$, implying V_i is independent in H. If $i \neq j$, then let $v \in V_i$ be arbitrary. As $Y_v \subseteq V_j$, v is adjacent to all other vertices in V_i and to the unique vertex in $V_j \setminus Y_v$. As $v \in V_i$

is arbitrary, $H[V_i] \cong K_a$ follows, and each vertex in V_i has a unique neighbor in V_j. Furthermore, fixing $v \in V_i$, for each vertex $z \in Y_v \subseteq V_j$ we have $v \in Y_z$, implying $Y_z \subseteq V_i$ and so $U_j = V_i$ by the arguments of the previous paragraph. Thus, it similarly follows that $H[V_j] \cong K_a$ and that each vertex in V_j has a unique neighbor in V_i. Consequently, the edges between V_i and V_j in H must form a matching, and $H[V_i \cup V_j] \cong H_0(a)$ then follows. □

Since $U_i \cup U_j \subseteq V_i \cup V_j$ implies $H = H[V_i \cup V_j] \vee H[V \setminus (V_i \cup V_j)]$ (as the set U_i contains all the non-neighbors of vertices in V_i), Claim 2 implies H is the join of t copies of $H_0(a)$, where $0 \leq t \leq \lfloor r/2 \rfloor$ since each copy has $2a$ vertices, and $r - 2t$ copies of $\mathcal{K}(a)$. Since $\mathcal{K}_{r-2t}(a)$ is the join of $(r-2t)$ copies of $\mathcal{K}(a)$, $H \cong J_r(a, t)$, and Theorem 3 follows. □

5 Conclusion

Our results have settled in the affirmative an open conjecture that complete bipartite graphs where the size of the partitions differs by at least two are $\mathcal{D}$-unique; that is, any two complete bipartite graphs where the size differs by more than one that have the same domination polynomial are isomorphic. Furthermore, we have extended these arguments to obtain a complete classification of $\mathcal{D}$-unique r-partite graphs, as well as the $\mathcal{D}$-equivalence class of complete equipartite graphs. However, our approach has shed little light on how to prove the conjecture (mentioned in Section 1 after Theorem 1) characterizing the $\mathcal{D}$-equivalence class of $K_{n,n+1}$, and we feel this is the next problem in this area worthy of attention.

Acknowledgements We would like to thank Ghodratollah Aalipour for providing us with preprints of [2] and [1].

References

1. Aalipour, G., Akbari, S., Ebrahimi, Z.: On D-equivalence class of $K_{m,n}$. Unpublished manuscript
2. Aalipour-Hafshejani, G., Akbari, S., Ebrahimi, Z.: On D-equivalence class of complete bipartite graphs. Ars Combinatoria, to appear (to appear)
3. Akbari, S., Alikhani, S., Peng, Y.H.: Characterization of graphs using domination polynomials. Eur. J. Comb. **31**(7), 1714–1724 (2010)
4. Alikhani, S.: Dominating sets and domination polynomials of graphs. Ph.D. thesis, University Putra Malaysia, 2009.
5. Alikhani, S., Peng, Y.H.: Dominating sets and domination polynomials of certain graphs, II. Opuscula Mathematica **30**(1), 37–51 (2010)
6. Alikhani, S., Peng, Y.H.: Domination polynomials of cubic graphs of order 10. Turkish Journal of Mathematics **35**(3), 355–366 (2011)
7. Alikhani, S., Peng, Y.H.: Introduction to domination polynomial of a graph. Ars Combinatoria **114**, 257–266 (2014)
8. Arocha, J.L., Llano, B.: Mean value for the matching and dominating polynomial. Discussiones Mathematicae Graph Theory **20**(1), 57–69 (2000)
9. Birkhoff, G.D.: A Determinant Formula for the Number of Ways of Coloring a Map. Annals of Mathematics **14**(1/4), 42–46 (1912)

10. Cameron, P.J.: Research problems from the BCC22. Discrete Mathematics **311**(13), 1074–1083 (2011)
11. Gutman, I., Harary, F.: Generalizations of the matching polynomial. Utilitas Mathematica **24**, 97–106 (1983)
12. Keevash, P.: Shadows and intersections: stability and new proofs. Advances in Mathematics **218**, 1685–1703 (2008)
13. Lovász, L.: Combinatorial Problems and Exercises. AMS Chelsea Publishing Series. AMS Chelsea Pub. (1993)
14. Turán, P.: On an extremal problem in graph theory (in Hungarian). Math. Fiz. Lapok **48**, 436–452 (1941)