Comparing Comparisons: Infinite Sums vs. Partial Sums

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The following problem originated during a classroom discussion:

Let C be the complex plane and let $D=\{z\in C\colon |z|<1\}$. Suppose that $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ are analytic in D. How does one compare two complex-valued functions? One common method is to use some type of "function measurement" or *functional*, F, that assigns real values $F(f)$ and $F(g)$ to f and g, respectively. In particular, we consider the following two conditions on f and g:

$$\text{(a)}\qquad \sum_{k=0}^{\infty}|b_k|^2r^{2k}\le\sum_{k=0}^{\infty}|a_k|^2r^{2k}\ \textit{for all}\ 0\le r<1.$$

$$\text{(b)}\qquad \sum_{k=0}^{n}|b_k|^2\le\sum_{k=0}^{n}|a_k|^2\ \textit{for all}\ n=0,1,2,\dots .$$

The following question arises: Which is the stronger condition, (a) or (b)? In other words, how do the two methods of comparison compare?

It is interesting to note that both conditions are known to hold in the special case that g is subordinate to f (i.e., $g(z)=f(w(z))$ for some w analytic in D satisfying $|w(z)|\le|z|$ for all $z\in D$). That is, we have the following:

LITTLEWOOD'S SUBORDINATION THEOREM. *If g is subordinate to f, then* (a) *holds.*

ROGOSINSKI'S THEOREM. *If g is subordinate to f, then* (b) *holds.*

Furthermore, an examination of the proof in Duren [1] of Rogosinski's Theorem reveals that Littlewood's Subordination Theorem is indirectly used. This observation might lead one to believe that, if a direct implication held, (a) would imply (b). Somewhat surprisingly, we have the following:

THEOREM. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ be analytic in D. If* (b) *holds then* (a) *holds. The converse is not true.*

Proof. Let f and g be analytic in D such that (b) is satisfied. Let $0\le r<1$ and let $A_n=\sum_{k=0}^{n}|a_k|^2$, $B_n=\sum_{k=0}^{n}|b_k|^2$. Since $r^{2k}\ge r^{2k+2}\ge\cdots\ge0$ and $B_k\le A_k$ for all $k\ge0$, we obtain (see [1] p. 193), using summation by parts, that

$$\begin{aligned}\sum_{k=0}^{n}|b_k|^2r^{2k}&=\sum_{k=0}^{n-1}(r^{2k}-r^{2k+2})B_k+r^{2n}B_n\\&\le\sum_{k=0}^{n-1}(r^{2k}-r^{2k+2})A_k+r^{2n}A_n=\sum_{k=0}^{n}|a_k|^2r^{2k}.\end{aligned}$$

Letting $n\to\infty$, we have that (a) holds.

To see that (a) does not imply (b), consider the following example: Let $f(z)=\sum_{k=0}^{\infty}z^{2k}$ and $g(z)=(1+\varepsilon)z+\sum_{k=1}^{\infty}z^{2k+1}$, where $\varepsilon>0$ is chosen so that $2\varepsilon+\varepsilon^2\le 1/2$. Define $I_2(r,f)=\sum_{k=0}^{\infty}|a_k|^2r^{2k}$. It follows that

$$I_2(r,f) = \sum_{k=0}^{\infty} r^{4k} = 1/(1-r^4) \text{ and}$$

$$\begin{aligned} I_2(r,g) &= (1+\varepsilon)^2 r^2 + \sum_{k=1}^{\infty} r^{4k+2} \\ &= (2\varepsilon + \varepsilon^2) r^2 + \sum_{k=0}^{\infty} r^{4k+2} \\ &\leq 1/2 + r^2/(1-r^4). \end{aligned}$$

Hence, $I_2(r,f) - I_2(r,g) \geq (1-r^2)/(1-r^4) - 1/2 = 1/(1+r^2) - 1/2 \geq 1/2 - 1/2 = 0$. Thus (a) holds. But when $n = 1$, the left-hand side of (b) is equal to $(1+\varepsilon)^2$ while the right-hand side of (b) is equal to 1. Thus, (b) is not satisfied for this f and g. This proves the theorem.

Indeed, given any prescribed nonnegative integer N, we can show that inequality (a) can hold while the inequality in (b) holds for n up to but not including $2N+1$. Let f be as above and choose $\varepsilon > 0$ such that $(1+\varepsilon)^{4N+2} - 1 \leq 1/2$. Define

$$g(z) = (1+\varepsilon)^{2N+1} z^{2N+1} + \sum_{k=0,\, k \neq N}^{\infty} z^{2k+1}.$$

$$\begin{aligned} \text{Now, } I_2(r,g) &= (1+\varepsilon)^{4N+2} r^{4N+2} + \sum_{k=0,\, k\neq N}^{\infty} r^{4k+2} \\ &= \left\{(1+\varepsilon)^{4N+2} - 1\right\} r^{4N+2} + \sum_{k=0}^{\infty} r^{4k+2} \\ &\leq 1/2 + r^2/(1-r^4). \end{aligned}$$

Thus as above, $I_2(r,g) \leq I_2(r,f)$ and so (a) holds. Also, the inequality in (b) is satisfied for all $n < 2N+1$. But for $n = 2N+1$, the left-hand side of (b) is equal to $N + (1+\varepsilon)^2$ while the right-hand side is equal to $N+1$. Therefore (b) is not satisfied.

Remark: One may also write $I_2(r,f) = 1/2\pi \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$, by Parseval's Identity [2].

REFERENCES

1. P. L. Duren, *Univalent Function Theory*, Springer-Verlag New York, Inc., New York, 1983.
2. W. Rudin, *Real and Complex Analysis*, 2nd edition, McGraw-Hill Book Co., New York, 1974.