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Series representations for special functions and mathematical constants

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Abstract We develop a collection of single-parameter series representations involving special functions and mathematical constants. The techniques used result in new representations as well as alternative proofs of known representations.

Keywords Series representations · Special functions · Mathematical constants

Mathematics Subject Classification 11A67 · 33B10 · 33B15 · 33B20 · 33B30 · 33C05 · 33E05

1 Introduction

The representation of a function as an infinite series plays an important role in various parts of mathematics as well as in physics and engineering. Over the years, numerous researchers published remarkable series representations for many special functions and offered interesting applications such as Apéry's proof that $\zeta(3)$ is irrational and Ramanujan's development of rapidly convergent series representations for the Riemann zeta function; we refer to [2,5–7,9,13,15], [16, Ch. 8–9], [17], [19, pp. 396–429], and the references therein for both recent and classical results. It is the aim of this paper to continue the work on this subject and to present new single-parameter series representations for the following classical functions:

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- Gamma and beta functions
- Trigonometric and inverse trigonometric functions
- Logarithm function
- Polylogarithm function
- Complete elliptic integrals
- Error function

As special cases of our results, we obtain series representations for various mathematical constants, like, for instance, $\arctan(1/\sqrt{2})$, $\log \frac{\sqrt{2}+1}{\sqrt{2}-1}$, and $1/\pi^{3/2}$.

In the next section we describe the technique to develop our series representations. In Sect. 3 we offer our representations for special functions, and in Sect. 4 we provide some additional series representations for the Catalan constant and π which are not derived from representations for special functions.

2 The basic theorem

The primary tool that we will use to develop single-parameter series representations is a technique we call *the λ-method* which was developed within the context of specific examples in [3,4] to find series representations from integrals of the form:

$$\int_0^1 \frac{g(x)}{(1-\alpha x^q)^p}\,\mathrm{d}x \tag{2.1}$$

with $\alpha = p = 1$. Here, we begin by summarizing the method in the form of a theorem, with a proof for completeness, and extend the technique to include other cases for α and p. The techniques used result in new representations as well as alternative proofs of known representations.

Theorem (The λ-method) *Suppose that the function G given by $G(x) = \dfrac{g(x)}{(1-\alpha x^q)^p}$ satisfies $g, G \in L^1[0,1]$, where $p \in \mathbb{R}$, $q > 0$, and $0 < \alpha \le 1$. With $\lambda < 1/2$ and*

$$b_k = b_k(\alpha, q) = \alpha^k \int_0^1 t^{kq} g(t)\,\mathrm{d}t \quad (0 \le k \in \mathbb{Z}),$$

it follows that

$$\int_0^1 \frac{g(x)}{(1-\alpha x^q)^p}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(p)_n}{n!(1-\lambda)^{n+p}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} b_k(\alpha, q), \tag{2.2}$$

where $(p)_n$ is the Pochhammer symbol given by $(p)_n = \frac{\Gamma(n+p)}{\Gamma(p)} = p(p+1)\cdots(p+n-1)$, $n \in \mathbb{N}$ and $(p)_0 = 1$.

Proof The case that $p = 0$ is trivial. Now suppose that $p > 0$, $q > 0$, $0 < \alpha \le 1$, and that the function G given by $G(x) = \dfrac{g(x)}{(1-\alpha x^q)^p}$ satisfies $g, G \in L^1[0,1]$. With $\lambda < 1/2$ and $x \in [0, 1)$, it follows that

$$\frac{|\alpha x^q - \lambda|}{1-\lambda} < 1$$

and hence

$$\frac{g(x)}{(1-\alpha x^q)^p} = \frac{g(x)}{(1-\lambda)^p \left(1 - \frac{\alpha x^q - \lambda}{1-\lambda}\right)^p} = \frac{g(x)}{(1-\lambda)^p} \sum_{n=0}^{\infty} \frac{(p)_n}{n!} \left(\frac{\alpha x^q - \lambda}{1-\lambda}\right)^n .$$

With

$$h_n(x) = \frac{g(x)(p)_n}{n!} \left(\frac{\alpha x^q - \lambda}{1-\lambda}\right)^n ,$$

we obtain

$$\begin{aligned}\left|\sum_{n=0}^{N} h_n(x)\right| &\leq \sum_{n=0}^{N} \frac{|g(x)|(p)_n}{n!} \left(\frac{|\alpha x^q - \lambda|}{1-\lambda}\right)^n \\ &= |g(x)| \left(\frac{1-\lambda}{1-\lambda-|\alpha x^q - \lambda|}\right)^p \\ &= |G(x)| \left(\frac{(1-\lambda)(1-\alpha x^q)}{1-\lambda-|\alpha x^q - \lambda|}\right)^p \leq M|G(x)|,\end{aligned}$$

where

$$M = \max\left\{\left(\frac{1-\lambda}{1-2\lambda}\right)^p , (1-\lambda)^p\right\}.$$

Since $G \in L^1[0, 1]$ and $\frac{1}{(1-\lambda)^p} \sum_{n=0}^{\infty} h_n(x)$ converges to $G(x)$ on $[0, 1)$, it follows by the Lebesgue Dominated Convergence Theorem that

$$\int_0^1 \sum_{n=0}^{\infty} h_n(x)\, \mathrm{d}x = \sum_{n=0}^{\infty} \int_0^1 h_n(x)\, \mathrm{d}x.$$

Therefore,

$$\begin{aligned}\int_0^1 \frac{g(x)}{(1-\alpha x^q)^p}\, \mathrm{d}x &= \int_0^1 \frac{g(x)}{(1-\lambda)^p} \sum_{n=0}^{\infty} \frac{(p)_n}{n!} \left(\frac{\alpha x^q - \lambda}{1-\lambda}\right)^n \mathrm{d}x \\ &= \frac{1}{(1-\lambda)^p} \sum_{n=0}^{\infty} \int_0^1 \frac{(p)_n}{n!} g(x) \left(\frac{\alpha x^q - \lambda}{1-\lambda}\right)^n \mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(p)_n}{n!(1-\lambda)^{n+p}} \int_0^1 g(x) \sum_{k=0}^{n} \binom{n}{k} (\alpha x^q)^k (-\lambda)^{n-k}\, \mathrm{d}x\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(p)_n}{n!(1-\lambda)^{n+p}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \int_0^1 g(x)(\alpha x^q)^k \, \mathrm{d}x$$
$$= \sum_{n=0}^{\infty} \frac{(p)_n}{n!(1-\lambda)^{n+p}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} b_k(\alpha, q).$$

In the case that $p < 0$, say $p = -\hat{p}$ where $\hat{p} > 0$, then

$$h_n(x) = \frac{g(x)(p)_n}{n!} \left(\frac{\alpha x^q - \lambda}{1-\lambda} \right)^n = g(x)(-1)^n \binom{\hat{p}}{n} \left(\frac{\alpha x^q - \lambda}{1-\lambda} \right)^n$$

and we find that

$$\left| \sum_{n=0}^{N} h_n(x) \right| \le \sum_{n=0}^{N} |g(x)| \binom{\hat{p}}{n} \left(\frac{|\alpha x^q - \lambda|}{1-\lambda} \right)^n$$
$$\le |g(x)| \sum_{n=0}^{\infty} \binom{\hat{p}}{n} \left(\frac{|\alpha x^q - \lambda|}{1-\lambda} \right)^n$$
$$= |g(x)| \left(1 + \frac{|\alpha x^q - \lambda|}{1-\lambda} \right)^{\hat{p}} \le |g(x)| 2^{\hat{p}}.$$

Since $g \in L^1[0,1]$ and $\frac{1}{(1-\lambda)^p} \sum_{n=0}^{\infty} h_n(x)$ converges to $G(x)$ on $[0, 1)$, the result follows as above. □

3 Series representations

In this section we apply the Theorem to obtain single-parameter series representations for some classical special functions.

3.1 Gamma and beta functions

Our first application makes use of an identity proved by Choi and Srivastava [12]:

$$\frac{\Gamma\left(1+\frac{s+1}{q}\right)\Gamma(1-p)}{(s+1)\Gamma\left(1+\frac{s+1}{q}-p\right)} = \int_0^1 \frac{x^s}{(1-x^q)^p} \mathrm{d}x \quad (q>0;\ s \ge 0;\ 0<p<1). \tag{3.1}$$

An extension of (3.1) is given in [18, Formula 8, p. 296]. With the help of (3.1), we are able to prove the following result.

Identity 3.1.1 *Let a, b, δ be real numbers with* $a > 0$, $0 < b < 1$, $\delta > 1/2$. *Then*

$$\frac{\delta^b \pi}{\sin(b\pi)} \frac{\Gamma(a)}{\Gamma(a+1-b)} = \sum_{n=0}^{\infty} \frac{\Gamma(n+b)}{n!\,\delta^n} \sum_{k=0}^{n} \binom{n}{k} \frac{(\delta-1)^{n-k}}{k+a}. \tag{3.2}$$

Proof Let $q \geq 1/a$. We set

$$s = qa - 1, \quad p = b, \quad \lambda = 1 - \delta.$$

Applying (3.1) and the Theorem with $\alpha = 1$, $g(t) = t^{qa-1}$ leads to

$$\frac{\Gamma\,(a+1)\;\Gamma\,(1-b)}{q\,a\,\Gamma(a+1-b)} = \int_0^1 \frac{t^{qa-1}}{(1-t^q)^b}\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(b)_n}{n!\,\delta^{n+b}} \sum_{k=0}^{n} \binom{n}{k} (\delta-1)^{n-k}\, b_k \tag{3.3}$$

with

$$b_k = \int_0^1 t^{kq} g(t)\mathrm{d}t = \frac{1}{q(k+a)}.$$

Next, we multiply both sides of (3.3) by $q\,\delta^b\,\Gamma(b)$ and apply the well-known identities

$$\Gamma(x+1) = x\Gamma(x) \quad \text{and} \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(x\pi)}.$$

This leads to (3.2). □

Special cases

(I) If we set $a = 1/3$, $b = 1/2$, $\delta = 1$ and make use of the formula

$$\Gamma(n+1/2) = \frac{\sqrt{\pi}}{2^n} \prod_{k=1}^{n} (2k-1),$$

then we obtain from (3.2) the representation:

$$\frac{\Gamma(1/3)}{\Gamma(5/6)} \sqrt{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n}{n!\,(n+1/3)}.$$

(II) We set $a = b = 1/3$, $\delta = 1$. Applying

$$\Gamma(n+1/3) = \frac{\Gamma(1/3)}{3^n} \prod_{k=1}^{n} (3k-2)$$

and (3.2) yields

$$\frac{2}{\sqrt{3}}\pi = \sum_{n=0}^{\infty} \frac{1}{n+1/3} \prod_{k=1}^{n} \left(1 - \frac{2}{3k}\right).$$

(III) Since

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} = \frac{2^{n+1}-1}{n+1},$$

we get from (3.2) with $a = 1$, $b = 1/2$, $\delta = 2$:

$$\sqrt{2} = \sum_{n=1}^{\infty} \frac{1-2^{-n}}{n-2^{-1}} \frac{(1/2)_n}{n!}.$$

Next, we study Euler's beta function. The representation

$$\frac{\sin((a-b)\pi/2)}{\sin((a+b)\pi/2)} B(a+1, b+1) = \int_0^1 \frac{x^a - x^b}{(1-x)^{a+b+2}} \mathrm{d}x \quad (a, b > -1;\ a+b < 0) \tag{3.4}$$

is given in [18, Formula 19, p. 297]. Applying (3.4), $(a)_n/n! = (-1)^n \binom{-a}{n}$, and the λ-method, we find

Identity 3.1.2 *Let* $x > 0$, $y > 0$ *and* $x + y < 2$. *Then for* $\lambda < 1/2$,

$$\frac{\sin((x-y)\pi/2)}{\sin((x+y)\pi/2)} B(x, y)$$
$$= (x-y) \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+x+y}} \binom{-x-y}{n} \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} \frac{(-1)^k}{(k+x)(k+y)}.$$

Special cases

(I) If we set $\lambda = 0$, then we find

$$\frac{\sin((x-y)\pi/2)}{\sin((x+y)\pi/2)} B(x, y) = (x-y) \sum_{n=0}^{\infty} \binom{-x-y}{n} \frac{(-1)^n}{(n+x)(n+y)}. \tag{3.5}$$

(II) We divide both sides of (3.5) by $(x - y)$ and let $x \to y$. Then for $x \in (0, 1)$,

$$B(x, x) = \frac{2}{\pi} \sin(x\pi) \sum_{n=0}^{\infty} \binom{-2x}{n} \frac{(-1)^n}{(n+x)^2}.$$

Setting $x = 1/3$ gives

$$\frac{4}{3\sqrt{3}\,\Gamma(2/3)^3}\pi^3 = \sum_{n=0}^{\infty}\binom{-2/3}{n}\frac{(-1)^n}{(n+1/3)^2}.$$

(III) If $x + y = 1$, then (3.5) yields the well-known formula:

$$\pi\cot(\pi x) = (1-2x)\sum_{n=0}^{\infty}\frac{1}{(n+x)(n+1-x)}.$$

An identity given by Choi [11, Eq. (1.4)] states that, for $a, b > -1$ and $m+1 \in \mathbb{N}$,

$$\frac{1}{2^{m+1}}\frac{\partial^m}{\partial a^m}B(a+1,b+1) = \int_0^{\pi/2}(\log\sin\theta)^m(\sin\theta)^{2a+1}(\cos\theta)^{2b+1}\,\mathrm{d}\theta. \tag{3.6}$$

The λ-method applied to (3.6) yields

Identity 3.1.3 *If $a, b > -1$, $m + 1 \in \mathbb{N}$ and $\lambda < 1/2$, then*

$$\frac{\partial^m}{\partial a^m}B(a+1,b+1) = (-1)^m\,m!\sum_{n=0}^{\infty}\frac{(-b)_n}{n!\,(1-\lambda)^{n-b}}\sum_{k=0}^{n}\binom{n}{k}\frac{(-\lambda)^{n-k}}{(a+k+1)^{m+1}}. \tag{3.7}$$

Proof Let $g(t) = t^{2a+1}(\log t)^m$. Since

$$\int_0^1 x^r(-\log x)^s\mathrm{d}x = \frac{\Gamma(s+1)}{(r+1)^{s+1}} \quad (r > -1;\ s > -1),$$

see [16, Formula 4.272 (6)], we obtain

$$b_k = \int_0^1 t^{2k}g(t)\,\mathrm{d}t = \frac{(-1)^m\,m!}{(2a+2k+2)^{m+1}} \quad (0 \le k \in \mathbb{Z}).$$

An application of the Theorem gives

$$\begin{aligned}&\int_0^{\pi/2}(\log\sin\theta)^m(\sin\theta)^{2a+1}(\cos\theta)^{2b+1}\,\mathrm{d}\theta\\ &= \int_0^1(\log t)^m\,t^{2a+1}(1-t^2)^b\,\mathrm{d}t\\ &= \frac{(-1)^m\,m!}{2^{m+1}}\sum_{n=0}^{\infty}\frac{(-b)_n}{n!\,(1-\lambda)^{n-b}}\sum_{k=0}^{n}\binom{n}{k}\frac{(-\lambda)^{n-k}}{(a+k+1)^{m+1}}.\end{aligned}$$

Combining this formula with (3.6) leads to (3.7). □

Special cases

(I) With $m = 1$, we find from (3.7):

$$B(a+1, b+1)\big(\psi(a+b+2) - \psi(a+1)\big) = \sum_{n=0}^{\infty} \frac{(-b)_n}{n!(1-\lambda)^{n-b}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\lambda)^{n-k}}{(a+k+1)^2},$$

where $\psi = \Gamma'/\Gamma$ denotes the digamma function.

(II) If $a = 0$, then

$$\frac{\gamma + \psi(b+2)}{b+1} = \sum_{n=0}^{\infty} \frac{(-b)_n}{n!\,(1-\lambda)^{n-b}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-\lambda)^{n-k}}{(k+1)^2}.$$

In particular, setting $\lambda = 0$, $b = 1/2$ yields the known result:

$$\log 2 = \frac{4}{3} - \frac{3}{4} \sum_{n=0}^{\infty} \frac{(-1/2)_n}{n!\,(n+1)^2} = \frac{4}{3} - \frac{3}{4} {}_3F_2(-1/2, 1, 1; 2, 2; 1);$$

see [19, Formula 57, p. 503].
Next, we make use of the integral representation:

$$B(a+1, b+1) = \int_0^1 \frac{x^a + x^b}{(1+x)^{a+b+2}} \mathrm{d}x \quad (a, b > -1)$$

which is published in [18, Formula 14, p. 297].

Identity 3.1.4 *For $x > 0$, $y > 0$ and $\mu < 0$, we have*

$$B(x, y) = \sum_{n=0}^{\infty} \frac{1}{(1-\mu)^{n+x+y}} \binom{-x-y}{n} \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \left(\frac{1}{k+x} + \frac{1}{k+y} \right). \tag{3.8}$$

Proof Using an argument similar to that in the proof of the λ-method, we proceed as follows (see [4] for a detailed description of the μ-method): Let $\mu < 0$ and $x \in [0, 1]$. Then

$$-1 < \frac{\mu + x}{1 - \mu} < 1.$$

Hence,

$$
\begin{aligned}
\frac{1}{(1+x)^{a+b+2}} &= (1-\mu)^{-a-b-2}\left(1+\frac{\mu+x}{1-\mu}\right)^{-a-b-2}\\
&= (1-\mu)^{-a-b-2}\sum_{n=0}^{\infty}\binom{-a-b-2}{n}\left(\frac{\mu+x}{1-\mu}\right)^{n}\\
&= \sum_{n=0}^{\infty}(1-\mu)^{-n-a-b-2}\binom{-a-b-2}{n}\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}x^{k}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
B(a+1,b+1) &= \int_0^1 (x^a+x^b)\sum_{n=0}^{\infty}(1-\mu)^{-n-a-b-2}\binom{-a-b-2}{n}\\
&\qquad \sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}x^{k}\mathrm{d}x\\
&= \sum_{n=0}^{\infty}(1-\mu)^{-n-a-b-2}\binom{-a-b-2}{n}\\
&\qquad \sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\left(\frac{1}{k+a+1}+\frac{1}{k+b+1}\right).
\end{aligned}
$$

We set $x = a+1 > 0$ and $y = b+1 > 0$ and obtain the desired representation. □

Special cases

(I) If $\mu \to 0$, then (3.8) leads to

$$
B(x,y) = \sum_{n=0}^{\infty}\binom{-x-y}{n}\left(\frac{1}{n+x}+\frac{1}{n+y}\right).
$$

Setting $x = y = 1/3$ yields

$$
\frac{2}{3\,\Gamma(2/3)^3}\pi^2 = \sum_{n=0}^{\infty}\binom{-2/3}{n}\frac{1}{n+1/3}.
$$

(II) Next, we set $\mu = -1$ in (3.8) and make use of the formula:

$$
\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+a} = \frac{\Gamma(a)\,n!}{\Gamma(n+1+a)} = \frac{(-1)^n}{(n+a)\binom{-a}{n}}.
$$

Then we obtain

$$2^{x+y}B(x,y)=\sum_{n=0}^{\infty}\frac{1}{2^n}\binom{-x-y}{n}\left(\frac{1}{(n+x)\binom{-x}{n}}+\frac{1}{(n+y)\binom{-y}{n}}\right).$$

3.2 Trigonometric and inverse trigonometric functions

In [18, Formula 12, p. 296] , we find

$$\frac{\pi}{\sin(a\pi)}=\int_0^1\frac{x^{a-1}}{(1-x)^a}\mathrm{d}x\quad(0<a<1).\tag{3.9}$$

An application of this formula and the Theorem implies

Identity 3.2.1 *For* $a\in(0,1)$ *and* $\lambda<1/2$,

$$\frac{\pi}{\sin(a\pi)}=\sum_{n=0}^{\infty}\frac{1}{(1-\lambda)^{n+a}}\binom{-a}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\lambda^{n-k}}{k+a}.$$

Special cases

(I) Setting $\lambda=0$ leads to

$$\frac{\pi}{\sin(a\pi)}=\sum_{n=0}^{\infty}\binom{-a}{n}\frac{(-1)^n}{n+a}.$$

(II) If we set $a=1/2$, $\lambda=-1$, then we arrive at

$$\frac{\pi}{\sqrt{2}}=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\binom{-1/2}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2k+1}.$$

Next we consider

$$\frac{-\mathrm{sgn}(y)}{\sqrt{y^2-1}}\arccos\left(-\frac{1}{y}\right)=\int_0^1\frac{1}{\sqrt{1-x^2}(x-y)}\mathrm{d}x\quad(|y|>1),$$

see [18, Formula 15, p. 299]. This leads to

Identity 3.2.2 *Let* $|y|>1$ *and* $\lambda<1/2$. *Then*

$$\frac{-\mathrm{sgn}(y)}{\sqrt{y^2-1}}\arccos\left(-\frac{1}{y}\right)=\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{1}{(1-\lambda)^{n+1/2}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\lambda^{n-k}J_{2k}(y)\tag{3.10}$$

with

$$J_k(y) = \int_0^1 \frac{x^k}{x-y} \mathrm{d}x = y^k \log\left(1 - \frac{1}{y}\right) + \sum_{j=1}^{k} \frac{y^{k-j}}{j} = -\sum_{j=k+1}^{\infty} \frac{y^{k-j}}{j}. \tag{3.11}$$

Special cases

(I) If we set $\lambda = 0$, then

$$\frac{-\operatorname{sgn}(y)}{\sqrt{y^2-1}} \arccos\left(-\frac{1}{y}\right) = \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} J_{2n}(y).$$

(II) Setting $y = 2$ and $y = -2$, respectively, yields

$$\frac{2}{3\sqrt{3}}\pi = \sum_{n=0}^{\infty} \left((-1)^n \binom{-1/2}{n} \sum_{j=2n+1}^{\infty} \frac{2^{2n-j}}{j} \right)$$

and

$$\frac{1}{3\sqrt{3}}\pi = \sum_{n=0}^{\infty} \left((-1)^n \binom{-1/2}{n} \sum_{j=2n+1}^{\infty} (-1)^{j-1} \frac{2^{2n-j}}{j} \right).$$

Using

$$-\frac{2}{\sqrt{y-1}} \arctan \frac{1}{\sqrt{y-1}} = \int_0^1 \frac{1}{\sqrt{1-x}} \frac{1}{x-y} \mathrm{d}x \quad (y > 1),$$

which is given in [18, Formula 14, p. 299], we obtain the following companion of (3.10).

Identity 3.2.3 *For $y > 1$ and $\lambda < 1/2$, it follows that*

$$-\frac{2}{\sqrt{y-1}} \arctan \frac{1}{\sqrt{y-1}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{1}{(1-\lambda)^{n+1/2}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \lambda^{n-k} J_k(y),$$

where $J_k(y)$ is given in (3.11).

Special cases

(I) We set $\lambda = 0$ and obtain

$$\frac{2}{\sqrt{y-1}} \arctan \frac{1}{\sqrt{y-1}} = \sum_{n=0}^{\infty} \left((-1)^n \binom{-1/2}{n} \sum_{j=n+1}^{\infty} \frac{y^{n-j}}{j} \right).$$

(II) The cases $y = 2$ and $y = 3$, respectively, yield

$$\frac{\pi}{2} = 2\arctan 1 = \sum_{n=0}^{\infty}\left((-2)^n \binom{-1/2}{n} \sum_{j=n+1}^{\infty} \frac{1}{j\,2^j}\right)$$

and

$$\arctan(1/\sqrt{2}) = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\left((-3)^n \binom{-1/2}{n} \sum_{j=n+1}^{\infty} \frac{1}{j\,3^j}\right).$$

We apply the well-known formula:

$$\frac{1}{2}\arcsin x + \frac{x}{2}\sqrt{1-x^2} = \int_0^x \sqrt{1-t^2}\mathrm{d}t = x\int_0^1 \sqrt{1-(xt)^2}\mathrm{d}t \quad (0 \le x \le 1)$$

and obtain

Identity 3.2.4 *Let* $0 \le x \le 1$ *and* $\lambda < 1/2$. *Then*

$$\frac{1}{2}\arcsin x + \frac{x}{2}\sqrt{1-x^2} = \sum_{n=0}^{\infty}\binom{1/2}{n}\frac{1}{(1-\lambda)^{n-1/2}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\lambda^{n-k}\frac{x^{2k+1}}{2k+1}.$$

We set $x = \sin y$ with $0 \le y \le \pi/2$. Then we find

$$\frac{1}{2}y + \frac{1}{4}\sin(2y) = \sum_{n=0}^{\infty}\binom{1/2}{n}\frac{1}{(1-\lambda)^{n-1/2}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\lambda^{n-k}\frac{(\sin y)^{2k+1}}{2k+1}.$$

Integration gives

$$\begin{aligned}\frac{\pi^2}{16} + \frac{1}{4} &= \int_0^{\pi/2}\left(\frac{1}{2}y + \frac{1}{4}\sin(2y)\right)\mathrm{d}y \\ &= \sum_{n=0}^{\infty}\binom{1/2}{n}\frac{1}{(1-\lambda)^{n-1/2}}\sum_{k=0}^{n}\binom{n}{k}\lambda^{n-k}\frac{(-4)^k}{(2k+1)^2\binom{2k}{k}}.\end{aligned}$$

The case $\lambda = 0$ leads to a result of Ewell [15]. Setting $\lambda = -1$ yields

$$\pi^2 = -4 + 16\sqrt{2}\sum_{n=0}^{\infty}\binom{1/2}{n}\left(-\frac{1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}\frac{4^k}{(2k+1)^2\binom{2k}{k}}.$$

3.3 Logarithm function

In [18, Formula 12, p. 299], we find

$$\sqrt{y}\log\frac{\sqrt{y}+1}{\sqrt{y}-1}=\int_0^1\frac{1}{\sqrt{1-x}}\frac{1}{\sqrt{1-x/y}}\mathrm{d}x\quad(y>1). \tag{3.12}$$

Identity 3.3.1 *Let $y > 1$ and $\lambda < 1/2$. Then we obtain*

$$\sqrt{y}\log\frac{\sqrt{y}+1}{\sqrt{y}-1}=\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{1}{(1-\lambda)^{n+1/2}}\sum_{k=0}^{n}\binom{n}{k}\lambda^{n-k}\frac{y^{-k}}{(k+1/2)\binom{-1/2}{k}}.$$

Proof The result follows from an application of the Theorem to the integral in (3.12) with $p = 1/2$, $q = 1$, $\alpha = 1/y$, $g(x) = 1/\sqrt{1-x}$, and

$$b_k = y^{-k}\int_0^1\frac{x^k}{\sqrt{1-x}}\mathrm{d}x = y^{-k}B(k+1,1/2)$$
$$= y^{-k}\frac{k!}{(1/2)_{k+1}}=\frac{(-1/y)^k}{(k+1/2)\binom{-1/2}{k}}.$$

□

Special case

Setting $\lambda = -1$, $y = 2$ yields

$$\log\frac{\sqrt{2}+1}{\sqrt{2}-1}=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\binom{-1/2}{n}\sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k\binom{n}{k}\frac{1}{(2k+1)\binom{-1/2}{k}}.$$

3.4 Polylogarithm function

In [14], the following integral formula for the polylogarithm function

$$\mathrm{Li}_s(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^s}\quad(s\in\mathbf{C};\ |z|<1)$$

is given:

$$\mathrm{Li}_s(z)=\frac{z}{\Gamma(s)}\int_0^1\frac{(-\log x)^{s-1}}{1-zx}\mathrm{d}x\quad(s>0;\ z\notin(1,\infty)); \tag{3.13}$$

see also [19, p. 762]. Applying (3.13) and the Theorem, we obtain

Identity 3.4.1 *Let $s > 0$, $|z| < 1$, and $\lambda < 1/2$. Then*

$$Li_s(z) = \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{z^{k+1}}{(k+1)^s}. \tag{3.14}$$

Special cases

(I) We have

$$\mathrm{Li}_2(\sqrt{5}-2) - \mathrm{Li}_2(-\sqrt{5}+2) = \frac{\pi^2}{12} - \frac{1}{6}\log^2(\sqrt{5}+2),$$

so that (3.14) with $s = 2$, $z = \pm(\sqrt{5}-2)$, $\lambda = -1$ leads to

$$\frac{\pi^2}{12} - \frac{1}{6}\log^2(\sqrt{5}+2) = \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(\sqrt{5}-2)^{2k+1}}{(2k+1)^2},$$

where $\lfloor\ \rfloor$ denotes the floor function. This representation is related to

$$\frac{\pi^2}{24} - \frac{1}{12}\log^2(\sqrt{5}+2) = \sum_{n=0}^{\infty} \frac{(\sqrt{5}-2)^{2n+1}}{(2n+1)^2}$$

which was published by Landen in 1780 and rediscovered by Ramanujan; see [8, pp. 41–42].

(II) Since

$$\mathrm{Li}_3(1/2) = \frac{1}{6}\log^3 2 - \frac{1}{12}\pi^2 \log 2 + \frac{7}{8}\zeta(3),$$

we conclude from (3.14) with $s = 3$, $z = 1/2$, $\lambda = -1$ that

$$\frac{2}{3}\log^3 2 - \frac{1}{3}\pi^2 \log 2 + \frac{7}{2}\zeta(3) = \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^k (k+1)^3}.$$

3.5 Complete elliptic integrals

For the complete elliptic integrals of the first and second kind, we have for $k \in (0, 1)$ the representations:

$$\mathcal{K}(k) = \int_0^1 \left(1-(kx)^2\right)^{-1/2} (1-x^2)^{-1/2} \mathrm{d}x$$

and

$$\mathcal{E}(k) = \int_0^1 \left(1-(kx)^2\right)^{1/2} (1-x^2)^{-1/2} \mathrm{d}x;$$

see [18, Formula 18, p. 299]). Using these formulas and the Theorem yields

Identity 3.5.1 *Let $k \in (0, 1)$ and $\lambda < 1/2$. Then*

$$\mathcal{K}(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1/2}} \binom{-1/2}{n} \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} \binom{-1/2}{j} k^{2j} \tag{3.15}$$

and

$$\mathcal{E}(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n-1/2}} \binom{1/2}{n} \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} \binom{-1/2}{j} k^{2j}. \tag{3.16}$$

Special cases

(I) Setting in (3.15) $\lambda = 0$ and $\lambda = -1$, respectively, we obtain

$$\mathcal{K}(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 k^{2n} = \frac{\pi}{2} {}_2F_1(1/2, 1/2; 1; k^2) \tag{3.17}$$

and

$$\mathcal{K}(k) = \frac{\pi}{2\sqrt{2}} \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \binom{-1/2}{n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \binom{-1/2}{j} k^{2j}. \tag{3.18}$$

We have

$$\mathcal{K}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) = \frac{\sqrt[4]{3}}{\sqrt[3]{128}} \frac{\Gamma(1/3)^3}{\pi}$$

so that (3.17) yields

$$\frac{1}{\pi^2} = c \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 \left(\frac{2-\sqrt{3}}{4}\right)^n$$

with

$$c = \frac{\sqrt[3]{16}}{\sqrt[4]{3}\,\Gamma(1/3)^3} = 0.09958....$$

In [19, Formula 176, p. 466], we find the identity:

$${}_2F_1(-n, 1/2; 1; z) = (1-z)^{n/2} P_n\left(\frac{2-z}{2\sqrt{1-z}}\right), \tag{3.19}$$

where P_n denotes the Legendre polynomial of the first kind. Setting $z = -1/2$ in (3.19) leads to

$$P_n\Big(\frac{5}{12}\sqrt{6}\Big) = \Big(\frac{2}{3}\Big)^{n/2} \sum_{j=0}^{n} \Big(-\frac{1}{2}\Big)^j \binom{n}{j} \binom{-1/2}{j}. \tag{3.20}$$

Using (3.20) and

$$\mathcal{K}\Big(\frac{1}{\sqrt{2}}\Big) = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}$$

we obtain from (3.18) with $k = 1/\sqrt{2}$:

$$\frac{1}{\pi^{3/2}} = \frac{\sqrt{2}}{\Gamma(1/4)^2} \sum_{n=0}^{\infty} \Big(-\frac{1}{2}\sqrt{\frac{3}{2}}\Big)^n \binom{-1/2}{n} P_n\Big(\frac{5}{12}\sqrt{6}\Big).$$

(II) Setting in (3.16) $\lambda = -1$ leads to

$$\mathcal{E}(k) = \frac{\pi}{\sqrt{2}} \sum_{n=0}^{\infty} \Big(-\frac{1}{2}\Big)^n \binom{1/2}{n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \binom{-1/2}{j} k^{2j}. \tag{3.21}$$

Applying

$$\mathcal{E}\Big(\frac{1}{\sqrt{2}}\Big) = \frac{\pi^{3/2}}{\Gamma(1/4)^2} + \frac{\Gamma(1/4)^2}{8\,\pi^{1/2}}$$

and (3.21) gives

$$\frac{\pi^{1/2}}{\Gamma(1/4)^2} + \frac{\Gamma(1/4)^2}{8\,\pi^{3/2}} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \Big(-\frac{1}{2}\sqrt{\frac{3}{2}}\Big)^n \binom{1/2}{n} P_n\Big(\frac{5}{12}\sqrt{6}\Big).$$

3.6 Error function

An application of the integral formula

$$\frac{\pi}{4} e^x [1 - \mathrm{erf}^2(\sqrt{x})] = \int_0^1 e^{-xt^2} \frac{1}{1+t^2}\, \mathrm{d}t \quad (x > 0),$$

see [1, Formula 7.4.12, p. 302], leads to

Identity 3.6.1 *Let $x > 0$ and $\mu < 0$. Then*

$$\frac{\pi}{2} e^x [1 - \mathrm{erf}^2(\sqrt{x})] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1-\mu)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k} \frac{\gamma(k+1/2, x)}{x^{k+1/2}}, \tag{3.22}$$

where

$$\gamma(a,x)=\int_0^x e^{-t}t^{a-1}\mathrm{d}t$$

denotes the incomplete gamma function.

Proof Using the method of proof, we have applied to establish Identity 3.1.4, we obtain for $\mu<0$:

$$\frac{1}{1+t^2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(1-\mu)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}t^{2k}\quad(|t|<1).$$

Thus,

$$\frac{\pi}{4}e^x[1-\mathrm{erf}^2(\sqrt{x})]=\sum_{n=0}^{\infty}\frac{(-1)^n}{(1-\mu)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\int_0^1 e^{-xt^2}t^{2k}\mathrm{d}t.$$

Since

$$\int_0^1 e^{-xt^2}t^{2k}\mathrm{d}t=\frac{1}{2\,x^{k+1/2}}\gamma(k+1/2,x),$$

we arrive at (3.22). □

Special case
If we set $x=1$ and let $\mu\to 0$, then we get

$$\frac{\pi e}{2}[1-\mathrm{erf}^2(1)]=\sum_{n=0}^{\infty}(-1)^n\gamma(n+1/2,1).$$

4 Concluding remarks

(I) We present two stand-alone series representations which are not derived as special cases of more general series representations of special functions.
Bradley [10] obtained for Catalan's constant,

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=0.91596...$$

the integral representations:

$$G=-\int_0^{\pi/4}\log(\tan x)\mathrm{d}x=\frac{1}{2}\int_0^1\frac{\arcsin t}{t\sqrt{1-t^2}}\mathrm{d}t.$$

Using the second integral, we find for $\lambda < 1/2$:

$$G = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{1}{(1-\lambda)^{n+1/2}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\lambda^{n-k}I_k,$$

where

$$I_0 = \frac{\pi}{2}\log 2 \quad \text{and} \quad I_k = \int_0^1 t^{2k-1}\arcsin t\,\mathrm{d}t = \frac{1}{4k}\Big(\pi - B(k+1/2, 1/2)\Big)$$
$$(k \geq 1).$$

The special case $\lambda = 0$ leads to

$$G = \frac{\pi}{4}\log 2 + \frac{\pi}{8}\sum_{n=1}^{\infty}\frac{1}{n}\binom{-1/2}{n}\left[(-1)^n - \binom{-1/2}{n}\right].$$

An application of

$$\frac{\pi}{16} = \int_0^1 \frac{x-1}{(x^2-2)(x^2-2x+2)}\mathrm{d}x$$

which is presented in [7], gives for $\lambda < 1/2$:

$$\frac{\pi}{2} = \sum_{n=0}^{\infty}\frac{1}{(1-\lambda)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{(-\lambda)^{n-k}}{2^k}\Delta_k,$$

where

$$\begin{aligned}\Delta_k = 4\int_0^1 \frac{(1-x)x^{2k}}{x^2-2x+2}\mathrm{d}x &= 4\int_0^{\pi/4}\tan t\,(1-\tan t)^{2k}\,\mathrm{d}t \\ &= 4\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j\int_0^{\pi/4}\tan^{j+1} t\,\mathrm{d}t \\ &= \sum_{j=0}^{2k}\binom{2k}{j}(-1)^j\left[\psi\Big(\frac{j}{4}+1\Big) - \psi\Big(\frac{j}{4}+\frac{1}{2}\Big)\right].\end{aligned}$$

In particular, setting $\lambda = 0$ yields

$$\frac{\pi}{2} = \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=0}^{2n}\binom{2n}{j}(-1)^j\left[\psi\Big(\frac{j}{4}+1\Big) - \psi\Big(\frac{j}{4}+\frac{1}{2}\Big)\right].$$

(II) The referee notes that novel series identities can be obtained by differentiation with respect to δ, λ, or μ in expressions such as (3.2), (3.10), or (3.22), respectively. We use the notations given in the Theorem and define

$$\begin{aligned} f(\lambda) &= (1-\lambda)^p \int_0^1 \frac{g(x)}{(1-\alpha x^q)^p} \mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(p)_n}{n!\,(1-\lambda)^n} \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} b_k(\alpha, q). \end{aligned}$$

Then for $0 \neq \lambda < 1/2$,

$$\begin{aligned} (\lambda-1)\, f'(\lambda) &= p(1-\lambda)^p \int_0^1 \frac{g(x)}{(1-\alpha x^q)^p} \mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(p)_n}{n!} \Big(\frac{\lambda}{\lambda-1}\Big)^n \sum_{k=0}^{n} \binom{n}{k} \frac{n-k+k\lambda}{(-\lambda)^{k+1}} b_k(\alpha, q). \end{aligned}$$

Setting $\lambda = -1$ for example, we get

$$p\, 2^p \int_0^1 \frac{g(x)}{(1-\alpha x^q)^p} \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(p)_n}{n!\, 2^n} \sum_{k=0}^{n} \binom{n}{k} (n-2k) b_k(\alpha, q).$$

Applying this to (3.9), we find for $a \in (0, 1)$:

$$\frac{a\pi}{\sin(a\pi)}\, 2^a = \sum_{n=0}^{\infty} \Big(-\frac{1}{2}\Big)^n \binom{-a}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{n-2k}{n+a}.$$

With $a = 1/4$, this yields

$$\frac{1}{2\sqrt[4]{2}}\,\pi = \sum_{n=0}^{\infty} \Big(-\frac{1}{2}\Big)^n \binom{-1/4}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{n-2k}{n+1/4}.$$

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