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On the modulus of the Grötzsch ring

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ABSTRACT

We study monotonicity, concavity and convexity properties of the function

$$G_r(\alpha) = \frac{\mu(r^\alpha)}{\alpha} \quad (0 < r < 1;\ \alpha > 0),$$

where μ denotes the modulus of the Grötzsch ring. Moreover, we apply our results to obtain various sharp functional inequalities.

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1. Introduction

The modulus of the Grötzsch extremal ring is defined for $r \in (0,1)$ as

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-r^2}\,)}{\mathcal{K}(r)},$$

where $\mathcal{K}$ is the complete elliptic integral of the first kind,

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2(t)}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\Big(\frac{(2n)!}{4^n\cdot(n!)^2}\Big)r^{2n}.$$

The function μ has remarkable applications in the theory of conformal mappings, in physics, and in number theory. The main properties of μ as well as interesting historical remarks and a detailed bibliography on this subject can be found in the excellent monograph "Conformal Invariants, Inequalities, and Quasiconformal Maps" by Anderson, Vamanamurthy and Vuorinen. In [4, Chapter 5] the following functional identities and

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bounds are presented:

$$\mu(r)\mu\Big(\sqrt{1-r^2}\Big)=\frac{\pi^2}{4},\quad \mu(r)\mu\Big(\frac{1-r}{1+r}\Big)=\frac{\pi^2}{2},\quad \mu(r)=2\mu\Big(\frac{2\sqrt{r}}{1+r}\Big),$$
$$\operatorname{arth}\sqrt[4]{r'}<\mu(r)<\frac{\pi^2}{4\operatorname{arth}\sqrt[4]{r}}\quad (r'=\sqrt{1-r^2}\,).$$

Moreover, many noteworthy monotonicity, convexity and concavity properties of functions which are defined in terms of μ are given in [4]. Applications of these results lead to elegant functional inequalities, like, for instance,

$$\mu(x)+\mu(y)\le\mu\Big(\frac{xy}{(1+\sqrt{1-x^2}\,)(1+\sqrt{1-y^2}\,)}\Big)\le 2\mu(\sqrt{xy}\,) \tag{1}$$

and

$$\sqrt{\mu(x)\mu(y)}\le\mu(\sqrt{xy}\,) \tag{2}$$

which are valid for $x,y\in(0,1)$. For extensions of (2) we refer to [7] and [9]. See also [8].

It is the aim of this note to continue the study of the Grötzsch ring function. Throughout, we use the following notation:

$$G_r(\alpha)=\frac{\mu(r^\alpha)}{\alpha},$$

where $r\in(0,1)$ and $\alpha>0$.

In Section 3, we offer monotonicity, convexity and concavity properties of this function and in Section 4 we use these results to obtain various functional inequalities. In order to prove our theorems we need some lemmas which we collect in Section 2. Finally, in Section 5, we state an open problem.

2. Lemmas

The following derivative formula is given in [3, Theorem 4.1(5)].

Lemma 1. *If $r\in(0,1)$, then*

$$\mu'(r)=-\frac{\pi^2}{4r(1-r^2)\mathcal{K}(r)^2}.$$

A proof for the next lemma can be found in [4, Theorem 3.30].

Lemma 2. *Let*

$$m(r)=\frac{2}{\pi}(1-r^2)\mathcal{K}(r)\mathcal{K}(\sqrt{1-r^2}\,). \tag{3}$$

(i) *The function $r\mapsto m(r)+\log r$ is strictly decreasing and concave from $(0,1)$ onto $(0,\log 4)$.*
(ii) *The function $r\mapsto -\frac{m(r)}{\log r}$ is strictly increasing from $(0,1)$ onto $(1,\infty)$.*

The last two lemmas provide functional inequalities for certain classes of functions. They are due to Petrović [6, p. 22] and Wright [11], respectively.

Lemma 3. *If $f : \mathbf{R} \to \mathbf{R}$ is strictly convex, then for $x, y > 0$ and $x, y < 0$,*

$$f(x) + f(y) < f(0) + f(x + y).$$

Lemma 4. *Let $I \subset \mathbf{R}$ be an interval. If $f : I \to \mathbf{R}$ is positive, monotone or convex, then, for $x, y, z \in I$,*

$$0 < (x - y)(x - z)f(x) + (y - x)(y - z)f(y) + (z - x)(z - y)f(z),$$

unless $x = y = z$.

3. Monotonicity, convexity and concavity

In this section, we study monotonicity, convexity and concavity properties of $G_r(\alpha)$ and $1/G_r(\alpha)$ as functions of α, where $r \in (0, 1)$ is a fixed parameter.

Theorem 1. *Let $r \in (0, 1)$. The function $\alpha \mapsto G_r(\alpha)$ is strictly decreasing and strictly log-convex from $(0, \infty)$ onto $(-\log r, \infty)$.*

Proof. Let $r \in (0, 1)$ and $\alpha > 0$. By differentiation we obtain

$$\alpha^2 \frac{d}{d\alpha} G_r(\alpha) = \mu'(r^\alpha) r^\alpha \alpha \log r - \mu(r^\alpha) = \mu'(t) t \log t - \mu(t)$$

with $t = r^\alpha \in (0, 1)$. Applying Lemma 1 we find

$$\alpha^2 \frac{d}{d\alpha} G_r(\alpha) = -\frac{\pi^2}{4(1 - t^2)\mathcal{K}(t)^2}[m(t) + \log t], \tag{4}$$

where m is defined in (3). From Lemma 2(ii) we conclude that the expression on the right-hand side of (4) is negative. This implies that $\alpha \mapsto G_r(\alpha)$ is strictly decreasing on $(0, \infty)$.

We have the representations

$$G_r(\alpha) = \frac{\pi}{2} \cdot \frac{-1}{\alpha \log(1 - r^\alpha)} \cdot \mathcal{K}(\sqrt{1 - r^{2\alpha}}) \cdot \frac{-\log(1 - r^\alpha)}{\mathcal{K}(r^\alpha)} \tag{5}$$

and

$$G_r(\alpha) = \pi \cdot (-\log r) \cdot \frac{\mathcal{K}(\sqrt{1 - r^{2\alpha}})}{-\log(r^{2\alpha})} \cdot \frac{1}{\mathcal{K}(r^\alpha)}. \tag{6}$$

Using the limit relations

$$\lim_{t \to 1-} \frac{\mathcal{K}(t)}{-\log(1 - t)} = \frac{1}{2} \quad \text{and} \quad \lim_{t \to 0+} \frac{\mathcal{K}(\sqrt{1 - t})}{-\log t} = \frac{1}{2}, \tag{7}$$

see [4, p. 49], we obtain from (5) and (6), respectively,

$$\lim_{\alpha \to 0+} G_r(\alpha) = \infty \quad \text{and} \quad \lim_{\alpha \to \infty} G_r(\alpha) = -\log r.$$

Next, we prove that $G_r(\alpha)$ is strictly log-convex with respect to α. We have

$$\frac{d}{d\alpha} \log G_r(\alpha) = \frac{\mu'(r^\alpha)}{\mu(r^\alpha)} r^\alpha \log r - \frac{1}{\alpha}.$$

We apply Lemma 1 and find

$$\frac{d}{d\alpha}\log G_r(\alpha) = -u(r^\alpha)\log r \tag{8}$$

with

$$u(t) = \frac{1}{m(t)}\Big(1 + \frac{m(t)}{\log t}\Big),$$

where m is given in (3). Lemma 2(i) reveals that the function $1/m$ is strictly increasing and positive on $(0,1)$. From Lemma 2(ii) we conclude that

$$t \mapsto -1 - \frac{m(t)}{\log t}$$

is strictly increasing and positive on $(0,1)$. It follows that u is strictly decreasing on $(0,1)$. Let $0 < \alpha < \beta$. Then, $0 < r^\beta < r^\alpha < 1$ and

$$-u(r^\alpha)\log r < -u(r^\beta)\log r. \tag{9}$$

Applying (8) and (9) we conclude that $\alpha \mapsto (d/d\alpha)\log G_r(\alpha)$ is strictly increasing on $(0,\infty)$. This means that $\alpha \mapsto G_r(\alpha)$ is strictly log-convex on $(0,\infty)$. □

Remark. A result presented in [4, Theorem 5.17] states that the function

$$g(s) = \frac{\mu(1/s)}{\log s}$$

is decreasing and convex on $(1,\infty)$. Applying Theorem 1 reveals that g is even strictly log-convex on $(1,\infty)$. Indeed, if $s > 1$, then we have

$$\begin{aligned} s^2\frac{d^2}{ds^2}\log g(s) &= s^2\frac{d^2}{ds^2}\log G_{1/e}(\log s) \\ &= \frac{d^2}{d\alpha^2}\log G_{1/e}(\alpha)\Big|_{\alpha=\log s} - \frac{d}{d\alpha}\log G_{1/e}(\alpha)\Big|_{\alpha=\log s} > 0. \end{aligned}$$

Theorem 2. *Let $r \in (0,1)$. The function $\alpha \mapsto 1/G_r(\alpha)$ is strictly concave on $(0,\infty)$.*

Proof. Let $r \in (0,1)$ and $\alpha > 0$. Then,

$$\frac{d}{d\alpha}\frac{1}{G_r(\alpha)} = \frac{\mu(r^\alpha) - \mu'(r^\alpha)\alpha r^\alpha \log r}{\mu(r^\alpha)^2} = \frac{\mu(t) - \mu'(t)t\log t}{\mu(t)^2},$$

where $t = r^\alpha \in (0,1)$. Let m be the function defined in (3). Applying Lemma 1 yields

$$\frac{d}{d\alpha}\frac{1}{G_r(\alpha)} = \frac{1}{\mu(t)}\Big(1 + \frac{\log t}{m(t)}\Big) = v(t), \quad \text{say.} \tag{10}$$

From Lemma 1 we conclude that $1/\mu$ is strictly increasing and positive on $(0,1)$ and Lemma 2(ii) reveals that

$$t \mapsto 1 + \frac{\log t}{m(t)}$$

is strictly increasing and positive on $(0,1)$. It follows that v is strictly increasing on $(0,1)$. Therefore, if $0<\alpha<\beta$, then

$$v(r^\beta) < v(r^\alpha).$$

Using (10) gives that $\alpha \mapsto (d/d\alpha)(1/G_r(\alpha))$ is strictly decreasing on $(0,\infty)$. Hence, $\alpha \mapsto 1/G_r(\alpha)$ is strictly concave on $(0,\infty)$. □

4. Inequalities

First, we offer sharp upper and lower bounds for the ratio $\mu(r^\alpha)/\mu(r^\beta)$ in terms of α and β.

Theorem 3. *Let α and β be real numbers with $\alpha>\beta>0$. Then we have for all $r\in(0,1)$:*

$$1 < \frac{\mu(r^\alpha)}{\mu(r^\beta)} < \frac{\alpha}{\beta}. \tag{11}$$

Both bounds are sharp.

Proof. Since μ is strictly decreasing on $(0,1)$, we conclude from $0<r^\alpha<r^\beta<1$ that the first inequality in (11) is valid. Applying Theorem 1 gives that the second inequality holds.

It remains to show that the bounds 1 and α/β are the best possible. Let

$$\Phi_{\alpha,\beta}(r) = \frac{\mu(r^\alpha)}{\mu(r^\beta)}.$$

Using the second limit relation given in (7) we obtain

$$\begin{aligned}\lim_{r\to 0+}\Phi_{\alpha,\beta}(r) &= \lim_{r\to 0+}\Big[\frac{\mathcal{K}(r^\beta)}{\mathcal{K}(r^\alpha)}\cdot\frac{\mathcal{K}(\sqrt{1-r^{2\alpha}})}{\mathcal{K}(\sqrt{1-r^{2\beta}})}\Big]\\ &= \lim_{r\to 0+}\frac{\mathcal{K}(\sqrt{1-r^{2\alpha}})}{\mathcal{K}(\sqrt{1-r^{2\beta}})} = \lim_{r\to 0+}\Big[\frac{\mathcal{K}(\sqrt{1-r^{2\alpha}})}{-\log(r^{2\alpha})}\cdot\frac{-\log(r^{2\beta})}{\mathcal{K}(\sqrt{1-r^{2\beta}})}\cdot\frac{\alpha}{\beta}\Big] = \frac{\alpha}{\beta}\end{aligned}$$

and applying the first relation in (7) leads to

$$\begin{aligned}\lim_{r\to 1-}\Phi_{\alpha,\beta}(r) &= \lim_{r\to 1-}\frac{\mathcal{K}(r^\beta)}{\mathcal{K}(r^\alpha)} = \lim_{r\to 1-}\Big[\frac{\mathcal{K}(r^\beta)}{-\log(1-r^\beta)}\cdot\frac{-\log(1-r^\alpha)}{\mathcal{K}(r^\alpha)}\cdot\frac{\log(1-r^\beta)}{\log(1-r^\alpha)}\Big]\\ &= \lim_{r\to 1-}\frac{\log(1-r^\beta)}{\log(1-r^\alpha)} = 1.\end{aligned}$$

Thus, both bounds given in (11) are sharp. □

The following Petrović-type inequality holds.

Theorem 4. *Let $r\in(0,1/e]$. For all real numbers $\alpha,\beta\in(0,1)$ and $\alpha,\beta>1$ we have*

$$\Big(\frac{\mu(r^\alpha)}{\alpha}\Big)^{1/\sqrt{\alpha}}\Big(\frac{\mu(r^\beta)}{\beta}\Big)^{1/\sqrt{\beta}} < \mu(r)\Big(\frac{\mu(r^{\alpha\beta})}{\alpha\beta}\Big)^{1/\sqrt{\alpha\beta}}. \tag{12}$$

The constant factor $\mu(r)$ is sharp.

Proof. Let $t \in \mathbf{R}$ and

$$\phi_r(t) = e^{t/2} w_r(e^{-t}) \quad \text{with} \quad w_r(\alpha) = \log G_r(\alpha).$$

Then,

$$\phi_r''(t) = \frac{1}{4} e^{t/2} w_r(e^{-t}) + e^{-3t/2} w_r''(e^{-t}).$$

From Theorem 1 we find

$$w_r(e^{-t}) > \log\log(1/r) \geq \log\log e = 0 \quad \text{and} \quad w_r''(e^{-t}) \geq 0.$$

This implies that ϕ_r is strictly convex on $\mathbf{R}$. Applying Lemma 3 yields for $s, t > 0$ and $s, t < 0$:

$$\phi_r(s) + \phi_r(t) < \phi_r(0) + \phi_r(s+t). \tag{13}$$

Setting

$$s = -\log\alpha \quad \text{and} \quad t = -\log\beta$$

we conclude from (13) that (12) is valid.

If we set $\alpha = \beta$ and let α tend to 1, then we obtain that in (12) the factor $\mu(r)$ cannot be replaced by a smaller constant. □

The next theorem offers an inequality with four variables. It is related to (12).

Theorem 5. *Let $r \in (0, 1/e]$. For all real numbers $\alpha, \beta, \gamma > 0$ we have*

$$1 \leq \left(\frac{\mu(r^\alpha)}{\alpha}\right)^{(\alpha-\beta)(\alpha-\gamma)} \left(\frac{\mu(r^\beta)}{\beta}\right)^{(\beta-\alpha)(\beta-\gamma)} \left(\frac{\mu(r^\gamma)}{\gamma}\right)^{(\gamma-\alpha)(\gamma-\beta)} \tag{14}$$

with equality if and only if $\alpha = \beta = \gamma$.

Proof. If $0 < r \leq 1/e$, then we obtain from Theorem 1 that the function

$$\alpha \mapsto \log G_r(\alpha)$$

is positive and strictly decreasing on $(0, \infty)$. Applying Lemma 4 with $f = \log G_r$ reveals that inequality (14) holds with equality if and only if $\alpha = \beta = \gamma$. □

The final two theorems provide counterparts of (1) and (2).

Theorem 6. *For all $x, y \in (0, 1)$ we have*

$$\mu(\sqrt{xy}\,) \leq \frac{1}{2} \frac{-\log(xy)}{\sqrt{\log x \cdot \log y}} \sqrt{\mu(x)\mu(y)}. \tag{15}$$

The constant factor $1/2$ is sharp. The sign of equality holds in (15) if and only if $x = y$.

Proof. Let $r \in (0,1)$. Using Theorem 1 gives that $\alpha \mapsto \log G_r(\alpha)$ is strictly convex on $(0,\infty)$. Thus, we obtain for $\alpha, \beta > 0$:

$$\log G_r\Big(\frac{\alpha+\beta}{2}\Big) \le \frac{1}{2}\big(\log G_r(\alpha) + \log G_r(\beta)\big) \tag{16}$$

with equality if and only if $\alpha = \beta$. We set

$$\alpha = \frac{\log x}{\log r} \quad \text{and} \quad \beta = \frac{\log y}{\log r}.$$

Then,

$$G_r(\alpha) = \frac{\log r}{\log x}\mu(x), \quad G_r(\beta) = \frac{\log r}{\log y}\mu(y), \quad G_r\Big(\frac{\alpha+\beta}{2}\Big) = \frac{2\log r}{\log(xy)}\mu(\sqrt{xy}\,). \tag{17}$$

From (16) and (17) we conclude that (15) is valid with equality if and only if $x = y$. □

Theorem 7. *For all $x, y \in (0,1)$ we have*

$$\frac{\log(xy)}{\mu(\sqrt{xy}\,)} \le \frac{\log x}{\mu(x)} + \frac{\log y}{\mu(y)}. \tag{18}$$

The sign of equality holds if and only if $x = y$.

Proof. Let $r \in (0,1)$. Applying Theorem 2 gives for $\alpha, \beta > 0$:

$$\frac{1}{2}\Big(\frac{1}{G_r(\alpha)} + \frac{1}{G_r(\beta)}\Big) \le \frac{1}{G_r\big(\frac{\alpha+\beta}{2}\big)} \tag{19}$$

with equality if and only if $\alpha = \beta$. Next, we set

$$\alpha = \frac{\log x}{\log r} \quad \text{and} \quad \beta = \frac{\log y}{\log r}.$$

Using (17) we conclude from (19) that (18) is valid. Moreover, the sign of equality is valid in (18) if and only if $x = y$. □

5. Open problem

A function $f : (0,\infty) \to \mathbf{R}$ is called completely monotonic if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \ge 0 \quad (x > 0;\ n = 0, 1, 2, \ldots). \tag{20}$$

Completely monotonic functions have remarkable applications in probability theory, numerical analysis, and other fields. A collection of the most important facts on completely monotonic functions is given in [10, Chapter IV]. In many recently published research papers it has been shown that numerous functions are completely monotonic. For example, $x \mapsto \Gamma(1+1/x)^x/x^a$ is completely monotonic if and only if $a \ge 0$ (as are other interesting relatives of Euler's gamma function). We refer to [1,2] and the references therein.

From (20) we conclude that a completely monotonic function is non-negative, decreasing and convex. Moreover, it is known that a completely monotonic function is log-convex; see [5]. Inspired by Theorem 1 we *conjecture* that for each $r \in (0,1)$ the function $\alpha \mapsto G_r(\alpha)$ is completely monotonic on $(0,\infty)$.

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