

Some Families of Fixed Points for the Eccentric Digraph Operator

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Abstract

We investigate the existence of fixed point families for the eccentric digraph (ED) operator, which was introduced in [1]. In [2], the notion of the period $\rho(G)$ of a digraph G (under the ED operator) was defined, and it was observed, but not proved, that for any odd positive integer m , $C_m \times C_m$ is periodic, and that $\rho(ED(C_m \times C_m)) = 2\rho(ED(C_m))$. Also in [2], the following question was posed: which digraphs are fixed points under the digraph operator? We provide a proof for the observations about $C_m \times C_m$, and in the process show that these products comprise a family of fixed points under ED . We then provide a number of other interesting examples of fixed point families.

1 Introduction

In [1], the idea of the eccentric digraph (ED) operator was introduced, and in [2], a number of conjectures were posed about iterating this operator. The *eccentricity* $e(u)$ of a vertex u in a digraph G is defined to be the maximum distance of any vertex from u . This maximum may be infinity, if some vertex is unreachable from u . The *eccentric digraph* $ED(G)$ of a digraph G has the same vertex set as G , and has a directed edge from u to v if and only if the distance from u to v in G is $e(u)$. In [2], the notion of the *period* $\rho(G)$ of a digraph G (under the ED operator) was defined, as follows: $\rho(G)$ is the smallest positive integer p such that $ED^{p+t}(G) = ED^t(G)$ for some nonnegative integer t . A digraph is said to be *periodic* if $t(G) = 0$. It was noted in [2] that $C_m \times C_m$ is periodic and $\rho(ED(C_m \times C_m)) = 2\rho(ED(C_m))$, where m is an odd positive integer, but the authors stated that they did not have a proof of this fact. Also in [2], the following question was posed: which digraphs are *fixed points* under the ED operator? That is, for which digraphs G do we have $ED(G) \cong G$, where \cong represents digraph isomorphism. Note that this question is posed in terms of isomorphism, rather than equality, because the only digraphs G for which $ED(G) = G$ are the complete

graphs K_n . In this paper, we provide a proof that, for any odd positive integer m , $C_m \times C_m$ is periodic, and we prove the noted equality $\rho(ED(C_m \times C_m)) = 2\rho(ED(C_m))$. In the process we will show that these products comprise a family of fixed points under ED , for odd m . We then provide a number of other interesting examples of fixed point families.

2 Cycle Products

Theorem 2.1 *Let $k \geq 0$, and let a_0, a_1, \dots, a_{2k} and b_0, b_1, \dots, b_{2k} be the vertices of two copies of C_{2k+1} . Then $ED(C_{2k+1} \times C_{2k+1})$ also has the structure of $C_{2k+1} \times C_{2k+1}$ and there is a graph isomorphism $\varphi : V(C_{2k+1} \times C_{2k+1}) \rightarrow V(ED(C_{2k+1} \times C_{2k+1}))$ defined by $\varphi((a_i, b_j)) = (a_{ki \oplus kj}, b_{(k+1)i \oplus kj})$, where \oplus denotes addition mod $2k + 1$.*

Proof. We will analyze the adjacencies of vertex (a_0, b_0) in $ED(C_{2k+1} \times C_{2k+1})$, and extend this analysis to all other vertices by symmetry. A breadth-first traversal in $C_{2k+1} \times C_{2k+1}$ easily reveals that the furthest vertices from (a_0, b_0) are the four unique vertices that have a shortest distance of $2k$ steps (by different paths) away from (a_0, b_0) , namely the set

$$\{(a_k, b_k), (a_{k+1}, b_k), (a_k, b_{k+1}), (a_{k+1}, b_{k+1})\}.$$

So in $ED(C_{2k+1} \times C_{2k+1})$, these four vertices (see Figure 1) will be the only ones adjacent to (a_0, b_0) . Furthermore, by symmetry, in $ED(C_{2k+1} \times C_{2k+1})$, each vertex (a_i, b_j) is adjacent to the four vertices

$$\{(a_{i \oplus k}, b_{j \oplus k}), (a_{i \oplus (k+1)}, b_{j \oplus k}), (a_{i \oplus k}, b_{j \oplus (k+1)}), (a_{i \oplus (k+1)}, b_{j \oplus (k+1)})\}.$$

Thus, every vertex has degree four, just as in the original product, and therefore it will suffice to show that the function φ is edge-preserving, and is a bijection.

To show that φ is edge-preserving, since every vertex in each graph has degree four, it will suffice to show that every horizontal edge is preserved under φ , as is every vertical edge. Given an arbitrary horizontal edge $\{(a_i, b_j), (a_i, b_{j \oplus 1})\}$ in the edge set of $C_{2k+1} \times C_{2k+1}$, φ maps the endpoints to $(a_{ki \oplus kj}, b_{(k+1)i \oplus kj})$ and $(a_{ki \oplus k(j \oplus 1)}, b_{(k+1)i \oplus k(j \oplus 1)})$ respectively. These two vertices are k steps apart in $C_{2k+1} \times C_{2k+1}$, since $k(j \oplus 1) \equiv kj \oplus k \pmod{2k+1}$, and hence are adjacent in $ED(C_{2k+1} \times C_{2k+1})$. A similar straightforward calculation verifies that every vertical edge is preserved under φ .

To show that φ is bijective, it will suffice to show that φ^2 is bijective, and for this it will suffice to show that φ^2 is injective, since φ^2 is a function from a finite set to itself. To this end we make the following calculation of φ^2 , where all operations on indices are reduced modulo $2k + 1$.

$$\begin{aligned}
& \varphi^2((a_i, b_j)) \\
&= \{\text{definition of } \varphi\} \\
& \varphi((a_{ki \oplus kj}, b_{(k+1)i \oplus kj})) \\
&= \{\text{definition of } \varphi\} \\
& (a_{k(ki \oplus kj) \oplus k((k+1)i \oplus kj)}, b_{(k+1)(ki \oplus kj) \oplus k((k+1)i \oplus kj)}) \\
&= \{\text{reduction modulo } 2k + 1\} \\
& (a_{(k+1)j}, b_{ki})
\end{aligned}$$

The proof that φ^2 is injective follows:

$$\begin{aligned}
& \varphi^2((a_i, b_j)) = \varphi^2((a_{i'}, b_{j'})) \\
& \iff \{\text{by the above calculation}\} \\
& (a_{(k+1)j}, b_{ki}) = (a_{(k+1)j'}, b_{ki'}) \\
& \iff \\
& (k+1)j = (k+1)j' \wedge ki = ki' \\
& \implies \{\text{multiplication by 2 and } 4k, \text{ respectively}\} \\
& 2(k+1)j = 2(k+1)j' \wedge 4k^2i = 4k^2i' \\
& \implies \{\text{since } 2(k+1) \equiv 4k \equiv 1 \pmod{(2k+1)}\} \\
& j = j' \wedge i = i' \\
& \iff \\
& (a_i, b_j) = (a_{i'}, b_{j'})
\end{aligned}$$

■

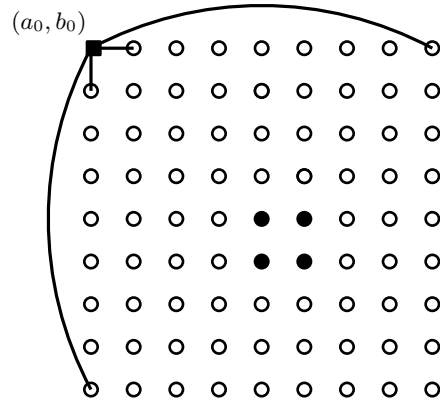


Figure 1: An illustration of Theorem 2.1 with $k = 4$, showing the vertices of $C_9 \times C_9$ in a grid structure. Each vertex is connected by four undirected edges (shown only for (a_0, b_0)) to its two horizontal neighbors and its two vertical neighbors, with wraparound from right to left and bottom to top, giving the grid a toroidal topology. The four filled vertices are those as far as possible (eight steps) from (a_0, b_0) , and so are adjacent to (a_0, b_0) in $ED(C_9 \times C_9)$.

The following lemma is implicit in the proof of a claim in Example 3.4 of [2]. It is stated and proved explicitly here for the convenience of the reader and for use in later theorems.

Lemma 2.2 *For all $k \geq 0$, $ED(C_{2k+1}) \cong C_{2k+1}$.*

Proof. This follows since two vertices in $ED(C_{2k+1})$ are adjacent in $ED(C_{2k+1})$ if and only if those vertices are exactly k steps apart in the cycle C_{2k+1} . The function $\varphi : V(C_{2k+1}) \rightarrow V(ED(C_{2k+1}))$, defined by $\varphi(v_i) = v_{i \otimes k}$, where \otimes denotes multiplication modulo $2k + 1$, is a graph isomorphism. It is easy to see that φ is a bijection, with $\varphi^{-1}(v_i) = v_{i \otimes 4k}$, since $4k \otimes k = 1$. Using modular arithmetic, it is straightforward to show that φ and φ^{-1} are edge-preserving. ■

Theorem 2.3 *For all $k \geq 0$, $ED^2(C_{2k+1} \times C_{2k+1}) = ED(C_{2k+1}) \times ED(C_{2k+1})$.*

Proof. First recall that two vertices in $ED(C_{2k+1})$ are adjacent in $ED(C_{2k+1})$ if those vertices are exactly k (or equivalently $k+1$) steps apart in the cycle C_{2k+1} . In the proof of Theorem 2.1 above, it was shown that $\varphi^2((a_i, b_j)) = (a_{(k+1)j}, b_{ki})$. This can be written as $\varphi^2 = \theta \circ \tau$, where $\theta((a_i, b_j)) = (a_{(k+1)i}, b_{kj})$ and $\tau((a_i, b_j)) = (a_j, b_i)$. Now observe that, in $C_{2k+1} \times C_{2k+1}$, every vertical edge $\{(a_i, b_j), (a_{i \oplus 1}, b_j)\}$ is mapped by θ to the vertices of a vertical edge $\{(a_{(k+1)i}, b_{kj}), (a_{(k+1)(i \oplus 1)}, b_{kj})\}$ of $ED(C_{2k+1}) \times ED(C_{2k+1})$, since $a_{(k+1)i}$ is $k + 1$ steps away from $a_{(k+1)(i \oplus (k+1))}$ in $ED(C_{2k+1})$. It is similarly straightforward to verify that every horizontal edge of $C_{2k+1} \times C_{2k+1}$ is mapped by θ to a horizontal edge of $ED(C_{2k+1}) \times ED(C_{2k+1})$. Therefore, θ is an isomorphism between $C_{2k+1} \times C_{2k+1}$ and $ED(C_{2k+1}) \times ED(C_{2k+1})$, and since τ is an automorphism of $C_{2k+1} \times C_{2k+1}$, the function φ^2 is also an isomorphism between $C_{2k+1} \times C_{2k+1}$ and $ED(C_{2k+1}) \times ED(C_{2k+1})$. Furthermore, by Theorem 2.1, φ^2 is an isomorphism between $C_{2k+1} \times C_{2k+1}$ and $ED^2(C_{2k+1} \times C_{2k+1})$. The proof can now be completed by observing that $ED^2(C_{2k+1} \times C_{2k+1}) = \varphi^2(C_{2k+1} \times C_{2k+1}) = ED(C_{2k+1}) \times ED(C_{2k+1})$. ■

Note that in the proof since τ is an automorphism of $C_{2k+1} \times C_{2k+1}$ that simply transposes the product structure, every horizontal edge of $C_{2k+1} \times C_{2k+1}$ is mapped under τ to a vertical edge of $ED(C_{2k+1}) \times ED(C_{2k+1})$ and every vertical edge is mapped to a horizontal edge. Furthermore, as noted in the proof, θ carries every vertical edge to another vertical edge, and every horizontal edge to another horizontal edge. Thus, overall $\varphi^2 = \theta \circ \tau$ carries every vertical edge to a horizontal edge, and every horizontal edge to a vertical edge (see Figure 2).

Theorem 2.4 *For all $k \geq 0$, and for all $t \geq 0$, $ED^{2t}(C_{2k+1} \times C_{2k+1}) = ED^t(C_{2k+1}) \times ED^t(C_{2k+1})$.*

Proof. The proof will be by induction on t , with the base case $t = 1$ provided by Theorem 2.3. In the calculation below of the inductive step, we replace the cycle C_{2k+1} by C to streamline the notation.

$$\begin{aligned}
& ED^{2t+2}(C \times C) \\
&= \\
& ED^{2t}(ED^2(C \times C)) \\
&= \{ \text{base case: Theorem 2.3} \} \\
& ED^{2t}(ED(C) \times ED(C)) \\
&= \{ \text{let } C' = ED(C) \text{ by Lemma 2.2} \} \\
& ED^{2t}(C' \times C') \\
&= \{ \text{inductive assumption} \} \\
& ED^t(C') \times ED^t(C') \\
&= \\
& ED^t(ED(C)) \times ED^t(ED(C)) \\
&= \\
& ED^{t+1}(C) \times ED^{t+1}(C)
\end{aligned}$$

■

Theorem 2.5 For all $k \geq 0$, $\rho(C_{2k+1} \times C_{2k+1}) = 2\rho(C_{2k+1})$.

Proof. By Theorem 2.4, and the definition of ρ , $\rho(C_{2k+1} \times C_{2k+1}) \leq 2\rho(C_{2k+1})$. In order to prove the reverse inequality, let s be an integer with $0 < s < 2\rho(C_{2k+1})$, and consider two cases. First, if s is even, then let $q = \frac{s}{2}$. In this case,

$$\begin{aligned}
& ED^s(C_{2k+1} \times C_{2k+1}) \\
&= \{ \text{by Theorem 2.1} \} \\
& \varphi^s(C_{2k+1} \times C_{2k+1}) \\
&= \{ s = 2q \} \\
& \varphi^{2q}(C_{2k+1} \times C_{2k+1}) \\
&= \{ \text{by Theorem 2.4} \} \\
& ED^q(C_{2k+1}) \times ED^q(C_{2k+1})
\end{aligned}$$

and since $q < \rho(C_{2k+1})$, $ED^q(C_{2k+1}) \neq ED(C_{2k+1})$. Therefore, $s \neq \rho(C_{2k+1} \times C_{2k+1})$. In the other case, if s is odd, then let $s = 2q + 1$, which gives $ED^s(C_{2k+1} \times C_{2k+1}) = \varphi(\varphi^{2q}(C_{2k+1} \times C_{2k+1}))$. After the proof of Theorem 2.3, it was noted that φ^2 carries every vertical edge to a horizontal edge, and by symmetry that φ^2 carries every horizontal edge to a vertical edge. Therefore for every q , φ^{2q} carries every horizontal edge to either a horizontal or a vertical edge, depending on the parity of q . Also note that φ carries each horizontal edge, and each vertical edge, to an edge joining two vertices that are in different factors of the domain product structure. Therefore $\varphi^s = \varphi(\varphi^{2q})$ cannot carry any horizontal (or by symmetry vertical) edge to itself, and therefore φ^s cannot be the identity map. Hence $s \neq \rho(C_{2k+1} \times C_{2k+1})$. ■

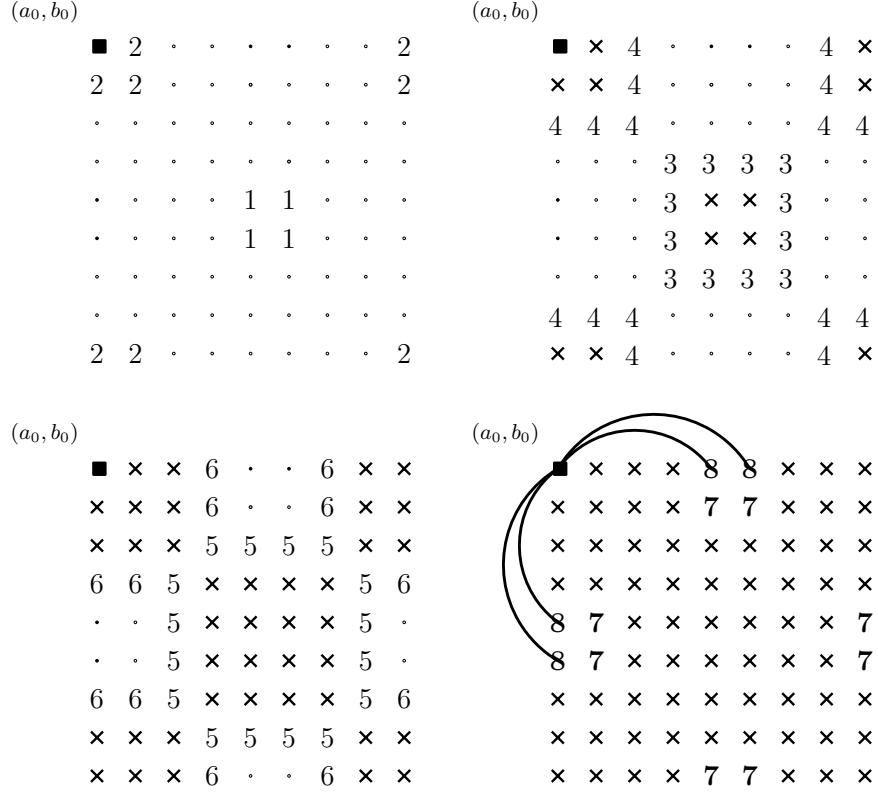


Figure 2: An illustration of Theorem 2.3 with $k = 4$. The first diagram shows the vertices that are one and two steps away from (a_0, b_0) in $ED(C_9 \times C_9)$ by annotating those vertices with a 1 and 2, respectively. The next diagram adds those that are 3 and 4 steps away, and the next adds those that are 5 and 6 steps away. The last diagram labels those that are 7 and 8 steps away, and those marked by an 8 are adjacent to (a_0, b_0) in $ED^2(C_9 \times C_9)$.

3 Conjunctions of cycles

Another way to combine two graphs G and H is by the *conjunction*, or *tensor product*, $G \wedge H$, as defined in [2]. In $G \wedge H$, the vertex set is the same as in the Cartesian product $G \times H$, namely it is the Cartesian product of the vertices of G and H . That is, $V(G \wedge H) = V(G) \times V(H)$. However, the edges are defined differently; there is an edge (in $G \wedge H$) between (a_1, b_1) and (a_2, b_2) if and only if there is an edge in G from a_1 to a_2 , and also an edge in H from b_1 to b_2 . The following theorem is straightforward, but we include a proof for the convenience of the reader.

Theorem 3.1 *For all $k \geq 0$, $C_{2k+1} \wedge C_{2k+1} \cong C_{2k+1} \times C_{2k+1}$.*

Proof. Define $\varphi : V(C_{2k+1} \times C_{2k+1}) \rightarrow V(C_{2k+1} \wedge C_{2k+1})$ by $\varphi(a_i, a_j) = (a_{i \oplus j}, a_{i \ominus j})$ where \oplus

and \ominus are addition modulo $2k + 1$ and subtraction modulo $2k + 1$, respectively. Since 2 has a multiplicative inverse (namely, $k + 1$) modulo $2k + 1$, it is easy to show by modular arithmetic that this function is an injection, and therefore is also a bijection. An explicit formula for φ^{-1} is given by $\varphi^{-1}(a_i, a_j) = (a_{(k+1)\otimes(i\oplus j)}, a_{(k+1)\otimes(i\ominus j)})$, where \otimes represents multiplication modulo $2k + 1$. It remains, then, to show that both φ and φ^{-1} are edge-preserving. Consider an edge $(a_i, a_j), (a_i, a_{j\oplus 1})$ in $C_{2k+1} \times C_{2k+1}$. The image under φ of these two vertices is the pair $((a_{i\oplus j}, a_{i\ominus j}), (a_{i\oplus j\oplus 1}, a_{i\ominus(j\oplus 1)}))$. It is easy to see that these two vertices are adjacent in $C_{2k+1} \wedge C_{2k+1}$, and the other three cases are similarly straightforward calculations. It is also easy to check with similar calculations that φ^{-1} is edge-preserving. ■

4 Fixed points

The following theorems address Question 3.3 in [2]. That is, which unlabeled graphs are fixed points under the ED operator? Here we consider unlabeled graphs since the only graphs that are fixed points as labeled graphs are the complete graphs. In the theorems and definitions below, any labeling of the vertices is described as an aid to exposition; these vertex labelings are not preserved in the isomorphisms used to exhibit the fixed point property.

Theorem 4.1 *For all $k \geq 0$, C_{2k+1} is a fixed point under the ED operator.*

Proof. This isomorphism is proved in Lemma 2.2. ■

Theorem 4.2 *For all $k \geq 0$, $C_{2k+1} \times C_{2k+1}$ is a fixed point under the ED operator.*

Proof. This isomorphism is proved in Theorem 2.1 above. ■

Theorem 4.3 *For all $k \geq 0$, $C_{2k+1} \wedge C_{2k+1}$ is a fixed point under the ED operator.*

Proof. This isomorphism follows from Theorem 3.1 and Theorem 4.2. ■

Theorem 4.4 *Let n be an odd positive integer, and let k be a positive integer such that k divides $n - 1$, and define $B_{n,k}$ to be the directed graph on vertices $\{v_0, v_1, \dots, v_{n-1}\}$, where each v_i has a directed edge to the “next- k ” vertices. That is, for each vertex v_i , there is a directed edge (v_i, v_j) for each $j = i \oplus 1, i \oplus 2, \dots, i \oplus k$, where \oplus denotes addition modulo n . Then $B_{n,k}$ is a fixed point under the ED operator.*

Proof. Starting at any vertex v_i , a breadth first search reveals that the vertices farthest away are the “previous- k ” vertices $\{v_{i\ominus 1}, v_{i\ominus 2}, \dots, v_{i\ominus k}\}$ (\ominus denotes subtraction modulo n), since k divides $n - 1$. So, in $ED(B_{n,k})$, there is a directed edge from v_i to each of these. We will show that the function $\varphi : V(B_{n,k}) \rightarrow V(ED(B_{n,k}))$ defined by $\varphi(v_i) = v_{n\ominus i}$ where \ominus represents subtraction modulo n , is a graph isomorphism. The function φ is clearly a bijection, with $\varphi = \varphi^{-1}$, so it will suffice to show that φ and φ^{-1} are edge-preserving. Let (v_i, v_j) be a directed edge of $B_{n,k}$, then $j = i \oplus m$ for some $m \in \{1, 2, \dots, k\}$. Therefore,

$$[n \ominus j = n \ominus (i \oplus m)] \iff [n \ominus j = (n \ominus i) \ominus m]$$

and so, in $ED(B_{n,k})$, there is a directed edge from $\varphi(v_i)$ to $\varphi(v_j)$, as required. A similar calculation (but starting with an edge in $ED(B_{n,k})$) verifies that φ^{-1} is edge-preserving. ■

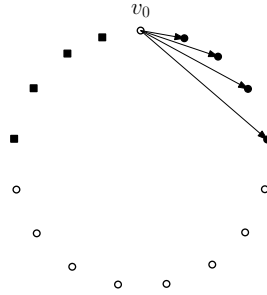


Figure 3: An illustration of Theorem 4.4 with $n = 17$ and $k = 4$. Each of the other vertices has the same pattern of adjacencies as v_0 , but these edges are not shown in the figure. The vertices adjacent from v_0 in $ED(B_{17,4})$ are marked with solid squares.

Theorem 4.5 *Let n , k , and s be positive integers, such that k divides $n-1$ and s is relatively prime to n . Define $B_{n,k,s}$ to be the digraph on vertices $\{v_0, v_1, \dots, v_{n-1}\}$, where each v_i has a directed edge to the “next- k with step size s ” vertices. That is, for each vertex v_i , there is a directed edge (v_i, v_j) for each $j = i \oplus s, i \oplus 2s, \dots, i \oplus ks$, where \oplus denotes addition modulo n . Then $B_{n,k,s}$ is a fixed point under the ED operator.*

Proof. The function $v_i \mapsto v_{si}$ is a digraph isomorphism between $B_{n,k}$ and $B_{n,k,s}$ since s and n are relatively prime. So by Theorem 4.4, $B_{n,k,s}$ is a fixed point under the ED operator. ■

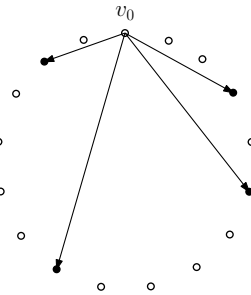


Figure 4: An illustration of Theorem 4.5 with $n = 17$ and $k = 4$, and $s = 5$. Note that the first three steps give v_5, v_{10} , and v_{15} , while the fourth step wraps around to v_3 . Each of the other vertices has the same pattern of adjacencies as v_0 , but these edges are not shown in the figure.

The next theorem refers to the *double complete digraph* (DK_n), the digraph on n vertices, in which there is a directed edge from every vertex to each of the others.

Theorem 4.6 *Let G be the digraph with $2n$ vertices formed by taking DK_n and adding n vertices each having a directed edge into every vertex of DK_n . Then G is a fixed point.*

Proof. Let v_1, \dots, v_n be the vertices of DK_n and u_1, \dots, u_n be the remaining vertices. Then, we will show there is a graph isomorphism $\varphi : V(G) \rightarrow V(ED(G))$ given by $\varphi(v_i) = u_i$ and $\varphi(u_i) = v_i$. The function is clearly a bijection on the vertices, so it will suffice to show that φ preserves the edge structure. Let (v_i, v_j) be one of the directed edges of G . Then, φ takes this edge to (u_i, u_j) which will certainly be in $ED(G)$ as there is no path from u_i to u_j in G . Similarly, the edges (u_i, v_j) in G will be sent to (v_i, u_j) by φ and again these edges are in $ED(G)$ as there is no path from v_j to u_i in G . Since all of the edges of G are of the form (v_i, v_j) or (u_i, v_j) and the edges of $ED(G)$ are only of the form (u_i, u_j) or (v_i, u_j) , we know φ is an isomorphism. Thus, G is a fixed point. ■

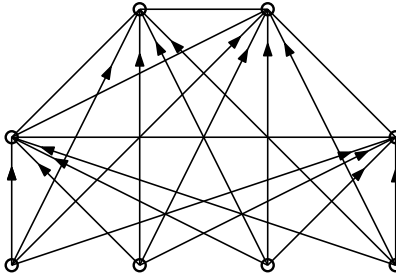


Figure 5: An illustration of Theorem 4.6 with $n = 4$. The upper four vertices comprise a complete graph, where each undirected edge denotes two edges, one in each direction. The lower four vertices each have four directed edges, one to each of the upper four vertices.

Another way to show that the digraphs of the family defined in Theorem 4.6 are all fixed points is to utilize Proposition 2.1 in [3], which is restated below as Proposition 4.7, for the convenience of the reader. This lemma relies on a construction G^- , the *reduction* of G , which is defined to be digraph G with all outgoing edges removed from any vertex v that is adjacent to all the other vertices. A digraph with no such vertices is said to be *reduced*, and the *reduced complement*, denoted $\overline{G^-}$, is defined to be the usual digraph complement of the reduction of G . Note that each digraph G in the family defined in Theorem 4.6 satisfies the local transitivity condition of 4.7, which implies that $ED(G) = \overline{G^-}$. So it suffices to verify that $G = \overline{G^-}$, which follows since G is reduced and isomorphic to its complement \overline{G} .

Proposition 4.7 Let G be a digraph of order $n > 1$. Then $ED(G) = \overline{G^-}$ if and only if for any vertex $u \in V(G)$ with eccentricity > 2 the following (local) transitive condition holds:

$$(u, v), (v, w) \in E(G) \Rightarrow (u, w) \in E(G), \forall v, w \in V(G) \text{ and } u \neq w.$$

Theorem 4.8 $ED(\overrightarrow{P_n})$ is a fixed point, for $n \geq 1$.

Proof. The cases $n = 1$ and $n = 2$ are easy to check separately. For $n \geq 3$, let $G = ED(\overrightarrow{P_n})$. Let v_0, \dots, v_{n-1} be the vertices along the unidirectional path. The edges of G consist of:

$$(v_0, v_{n-1}), \text{ and} \\ (v_i, v_j) \text{ for all } 0 \leq j < i \leq n - 1$$

The edges of $ED(G)$ consist of:

$$(v_{n-2}, v_{n-1}), \\ (v_i, v_j) \text{ for all } 0 \leq i < j \leq n - 2, \text{ and} \\ (v_{n-1}, v_j) \text{ for all } 0 \leq j \leq n - 2$$

Using these lists of edges, it is easy to see there is a digraph isomorphism between G and $ED(G)$ given by $\varphi : V(G) \rightarrow V(ED(G))$ with $\varphi(v_{n-1}) = v_{n-1}$ and $\varphi(v_i) = v_{n-2-i}$ for $0 \leq i \leq n - 2$. Thus, $G = ED(\overrightarrow{P_n})$ is a fixed point. ■

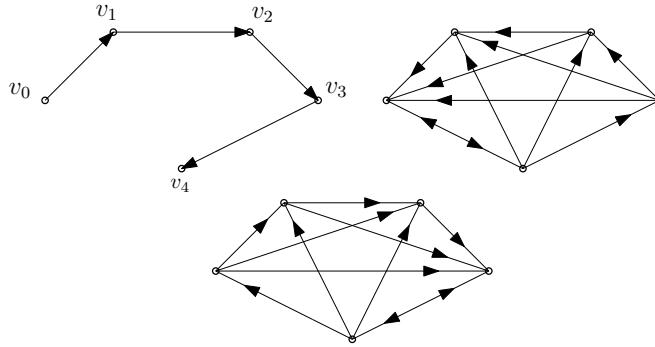


Figure 6: An illustration of Theorem 4.8 with $n = 5$. The upper left digraph is $\overrightarrow{P_5}$, the upper right digraph is $ED(\overrightarrow{P_5})$, and the lower digraph is $ED^2(\overrightarrow{P_5})$.

The last few fixed point families we will describe are all based on the following balanced coloring property of any initial segment of positive integers of odd length.

Lemma 4.9 *Let n be an odd positive integer, and let k be a positive integer smaller than n . For each pair $\{i, j\}$ of positive integers with $i + j = n$, color one red and the other blue. Then either k is red, or there is a pair (a, b) of red numbers, such that $a \oplus b = k$, where \oplus represents addition modulo n .*

Proof. If there is no pair (a, b) with $a \neq b$ and $a + b = k$, then a bipartite graph can be constructed in which every vertex has degree 2 except vertex k and vertex $k \oslash 2$. Here $k \oslash 2$ represents division modulo n (Note that since n is odd, every number k smaller than n has a unique half modulo n). Construct the graph by beginning with k , then attaching k to $n - k$, then attaching $n - k$ to $k \ominus (n - k)$, and so on, alternating as in Figure 7 until arriving at $k \oslash 2$. At this point, k and $k \oslash 2$ must be in opposite parts of the partition, since the values in the same part as k are of the form $n - j$ for some j ($\neq n - j$) in the opposite part. This is not possible for $k \oslash 2$, since $n - k \oslash 2 = k \oslash 2$. Therefore if k is blue, then $k \oslash 2$ is red, so taking $a = k \oslash 2$, and $b = k \oslash 2$, we have $a \oplus b = 2(k \oslash 2) = k$. ■

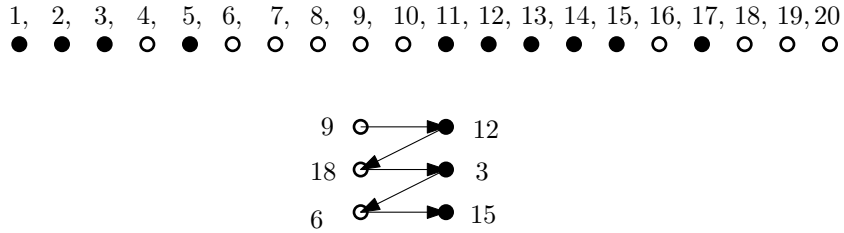


Figure 7: An illustration of Lemma 4.9 with $n = 21$ and $k = 9$, and $k \oslash 2 = 15$. A solid dot represents the color red, and an open dot represents the color blue.

Theorem 4.10 *Let n be an odd positive integer, and let R be a subset of $\{1, 2, \dots, n - 1\}$ such that for every pair $\{i, j\}$ of positive integers with $i + j = n$, exactly one of $\{i, j\}$ is a member of R . Define $G_{n,R}$ to be the directed graph on vertices $\{v_0, v_1, \dots, v_{n-1}\}$, where each v_i has a directed edge to vertex $v_{i \oplus k}$ if and only if $k \in R$, where \oplus denotes addition modulo n . Then $G_{n,R}$ is a fixed point under the ED operator.*

Proof. By Lemma 4.9, for every vertex v_i , the vertices v_j that are not adjacent from v_i are all a distance of two steps away from v_i , which makes them the vertices adjacent from v_i in the eccentric digraph $ED(G_{n,R})$. Therefore, $ED(G_{n,R}) = G_{n,\bar{R}}$, where \bar{R} denotes the complement of R in $\{1, 2, \dots, n - 1\}$. We will show that the function $\varphi : V(G_{n,R}) \rightarrow V(G_{n,\bar{R}})$ defined by $\varphi(v_i) = v_{n \ominus i}$, where \ominus represents subtraction modulo n , is a graph isomorphism. The function is clearly a bijection on the vertices, so it will suffice to show that φ and φ^{-1} are edge-preserving. Let (v_i, v_j) be a directed edge of $G_{n,R}$, then $j = i \oplus k$ for some $k \in R$. Therefore, $n \ominus j = (n \ominus i) \oplus (n \ominus k)$, and $n - k \in \bar{R}$, and so in $G_{n,\bar{R}}$, there is a directed edge from $\varphi(v_i)$ to $\varphi(v_j)$, as required. The verification that φ^{-1} is edge-preserving is similarly straightforward. ■

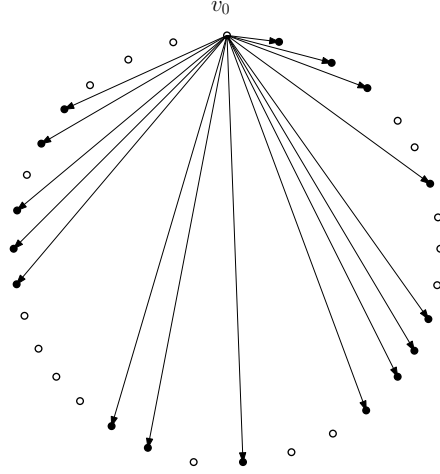


Figure 8: An example of the digraph $G_{n,R}$ of Theorem 4.10 with $n = 33$ and $R = \{1, 2, 3, 6, 10, 11, 12, 13, 16, 18, 19, 24, 25, 26, 28, 29\}$. Each of the other vertices has the same pattern of adjacencies as v_0 , but these edges are not shown in the figure.

This is another family of digraphs which are interesting to examine in the light of Proposition 2.1 in [3], which is included above as Proposition 4.7. The proof of Theorem 4.10 can be modified slightly into a proof that a digraph G satisfying the conditions of Theorem 4.10 is isomorphic to its reduced complement $\overline{G^-}$. Furthermore, Lemma 4.9 can be used to show the local transitivity property of Lemma 4.7. Therefore from Proposition 2.1 in [3], it can be shown that G is a fixed point under the ED operator.

Corollary 4.11 *Let n be an odd positive integer, and let k be a positive integer such that k divides $n - 1$, and such that $(n - 1)/k$ is even. Define $G_{n,k}$ to be the digraph on vertices $\{v_0, v_1, \dots, v_{n-1}\}$, where each v_i has a directed edge to vertex $v_{i \oplus j}$ if and only if $(j - 1)/k$ is even. Then $G_{n,k}$ is a fixed point under the ED operator.*

Proof. $G_{n,k}$ satisfies the conditions of Theorem 4.10, by taking R to be the set of positive integers j with $(j - 1)/k$ even. With this choice of R , for each $j \in R$, $(n - j - 1)/k + (j - 1)/k = (n - 2)/k$, which is odd. Therefore, $(n - j - 1)/k$ must be odd, which implies that $n - j \notin R$. This confirms that set R satisfies the conditions of Theorem 4.10, and therefore $G_{n,k}$ is a fixed point under the ED operator. ■

Corollary 4.12 *Let $P = (k_1, k_2, \dots, k_m)$ be a palindromic sequence of positive integers, so that $k_u = k_v$ whenever $u + v = m + 1$, and let $n = 1 + 2 \sum_{i=1}^m k_i$. Define $G_{n,P}$ to be the directed graph on vertices $\{v_0, v_1, \dots, v_{n-1}\}$, where each v_i has a directed edge to vertex $v_{i \oplus j}$ if and only if $2 \sum_{i=1}^q k_i < j \leq (2 \sum_{i=1}^q k_i) + k_{q+1}$ for some q with $0 \leq q < m$. Then $G_{n,P}$ is a fixed point under the ED operator.*

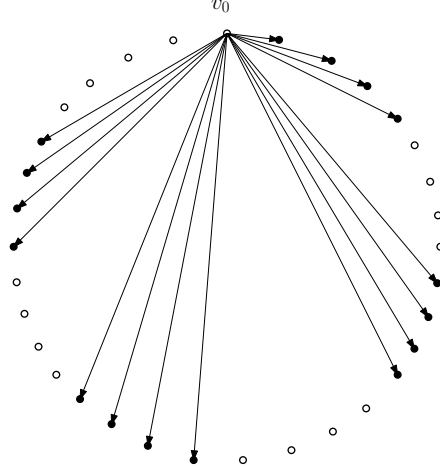


Figure 9: A $G_{n,k}$ of Corollary 4.11 with $n = 33$ and $k = 4$. Each of the other vertices has the same pattern of adjacencies as v_0 , but these edges are not shown in the figure.

Proof. Apply Theorem 4.10, where

$$R = \{j | 2 \sum_{i=1}^q k_i < j \leq (2 \sum_{i=1}^q k_i) + k_{q+1} \text{ for some } q \text{ with } 0 \leq q < m\}.$$

The complement of this set can be denoted by

$$\bar{R} = \{h | (2 \sum_{i=1}^q k_i) + k_{q+1} < h \leq 2 \sum_{i=1}^{q+1} k_i \text{ for some } q \text{ with } 0 \leq q < m\}$$

Then subtracting each part of this inequality from $n - 1$ gives:

$$\begin{aligned} (n-1) - 2 \sum_{i=1}^q k_i &> (n-1) - j \geq (n-1) - ((2 \sum_{i=1}^q k_i) + k_{q+1}) \\ &\iff \\ 2 \sum_{i=1}^m k_i - 2 \sum_{i=1}^q k_i &> n-1-j \geq 2 \sum_{i=1}^m k_i - (2 \sum_{i=1}^q k_i) - k_{q+1} \\ &\iff \\ 2 \sum_{i=q+1}^m k_i &> n-1-j \geq (2 \sum_{i=q+1}^m k_i) - k_{q+1} \\ &\iff \\ 2 \sum_{i=1}^{m-q} k_i &> n-1-j \geq (2 \sum_{i=1}^{m-q} k_i) - k_{m-q} \\ &\iff \\ (2 \sum_{i=1}^{m-q} k_i) - k_{m-q} &\leq n-1-j < 2 \sum_{i=1}^{m-q} k_i \\ &\iff \\ (2 \sum_{i=1}^{m-q-1} k_i) + k_{m-q} &\leq n-1-j < 2 \sum_{i=1}^{m-q} k_i \\ &\iff \\ (2 \sum_{i=1}^{m-q-1} k_i) + k_{m-q} &< n-j \leq 2 \sum_{i=1}^{m-q} k_i \end{aligned}$$

This shows that for each $j \in R$, $n - j \in \overline{R}$, which satisfies the condition for set R in Theorem 4.10, and therefore $G_{n,P}$ is a fixed point under the ED operator. ■

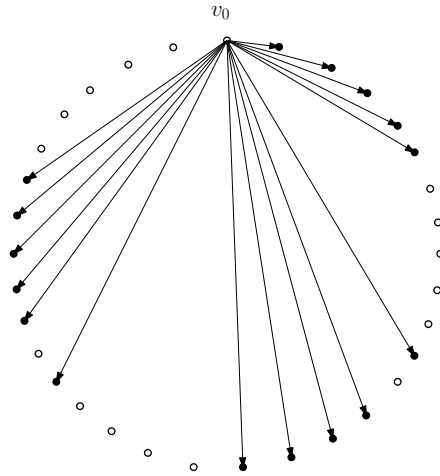


Figure 10: A $G_{n,R}$ of Corollary 4.12 with $n = 33$ and $P = \{5, 1, 4, 1, 5\}$. Each of the other vertices has the same pattern of adjacencies as v_0 , but these are not shown in the figure.

References

- [1] F. Buckley, The eccentric digraph of a graph, Congr. Numer. 149 (2001), 65-76.
- [2] J. Gimbert, M. Miller, F. Ruskey and J. Ryan, Iterations of eccentric digraphs, Bull. Inst. Combin. Appl. 45 (2005), 41-50.
- [3] J. Gimbert, N. Lopez, M. Miller and J. Ryan, Characterization of eccentric digraphs, Discrete Mathematics Vol. 306, Issue 2 (2006), 210-219.