On Inequalities for ${}_2F_1$ and Related Means

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Abstract

This informal working paper provides an expository survey of some inequalities and associated conjectures involving the Gaussian hypergeometric function ${}_2F_1$ and closely related bivariate means. Recent as well as previously established results are presented for which the conjectures are known to hold.

This basic investigation begins with the fundamental concept of a bivariate *mean* which is defined as a continuous function $\mathcal{M} : (0,\infty) \times (0,\infty) \to \mathbb{R}$ satisfying $\min\{x,y\} \leq \mathcal{M}(x,y) \leq \max\{x,y\}$ for all $x, y \geq 0$. Desirable properties possessed by some means include *strictness*: $\mathcal{M}(x,y) = x$ or $\mathcal{M}(x,y) = y$ if and only if $x = y$; *symmetry*: $\mathcal{M}(x,y) = \mathcal{M}(y,x)$; and *homogeneity*: $\mathcal{M}(\lambda x, \lambda y) = \lambda \mathcal{M}(x,y)$ for $\lambda > 0$. For the purpose of investigating inequalities involving a homogeneous mean $\mathcal{M}$, it suffices to consider the form $\mathcal{M}(1,r)$. Passing back to the more general case is then accomplished by using $x\mathcal{M}(1,y/x) = \mathcal{M}(x,y)$.

Simple examples of means abound. The arithmetic mean $\mathcal{A}(x,y) \equiv \frac{x+y}{2}$ and the geometric mean $\mathcal{G}(x,y) \equiv \sqrt{xy}$ are two of the most famous (homgeneous, symmetric, and strict) means. These two are in fact special cases of the family of power means given by

$$\mathcal{A}_\lambda(x,y) \equiv \left(\frac{x^\lambda + y^\lambda}{2}\right)^{1/\lambda} \qquad (\lambda \neq 0),$$

with $\mathcal{A}_0(x,y) \equiv \sqrt{xy}$. A standard argument can be used to show that the function $\lambda \mapsto \mathcal{A}_\lambda$ is increasing. From this follows one of many proofs (see [9]) of the well-known arithmetic mean - geometric mean inequality:

$$\mathcal{G} = \mathcal{A}_0 \leq \mathcal{A}_1 = \mathcal{A}$$

where each is evaluated at (x,y).

Other interesting means (but perhaps less familiar) include the *logarithmic mean* $\mathcal{L}(x,y) \equiv (x-y)/(\ln x - \ln y)$ and the *identric mean* $\mathcal{I}(x,y) \equiv \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{1/(x-y)}$ (with $\mathcal{L}(x,x) \equiv x \equiv \mathcal{I}(x,x)$ preserving continuity). The previous inequality has been refined (e.g. see [9]) by these two means to yield

$$\mathcal{G} \leq \mathcal{L} \leq \mathcal{I} \leq \mathcal{A}.$$

This inequality chain can again be refined (e.g. see [9]) by the established fact that

$$\mathcal{L} \leq \mathcal{A}_{1/3} \tag{1}$$

and

$$\mathcal{A}_{2/3} \leq \mathcal{I}. \tag{2}$$

The power mean orders 1/3 and 2/3 are sharp (i.e. best possible) in (1) and (2) respectively. It is worth noting that (1) and (2) have been recently generalized by Theorem C below (see also [13]).

Means also arise naturally from such familiar calculus concepts as the arc length of an ellipse. Given an ellipse parametrized by $(x\sin t, y\cos t)$, with $x \geq y > 0$, its arc length is given by $4E(x,y)$ where

$$E(x,y) \equiv \int_0^{\pi/2} \sqrt{x^2\cos^2 t + y^2 \sin^2 t}\, dt.$$

It follows that $\hat{E}(x,y) \equiv \frac{2}{\pi}E(x,y)$ is a mean since it is continuous and satisfies $y \leq \hat{E}(x,y) \leq x$ (for $0 < y \leq x$). $\hat{E}$ is also homgeneous, symmetric, and strict. Moreover, the function $\mathcal{E}(r) \equiv E(1, \sqrt{1-r^2})$, called the *complete elliptic integral* of the second kind, is a very important and much-studied function arising in mathematical physics (see [14]). It turns out that $\mathcal{E}$ can be expressed as a special case of the Gaussian hypergeometric function defined by

$${}_2F_1(\alpha,\beta;\gamma;z) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n, \quad |z| < 1,$$

and $(\alpha)_n \equiv \Gamma(\alpha+n)/\Gamma(\alpha) = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n \in \mathbb{N}$ and $(\alpha)_0 \equiv 1$. Here

$$\Gamma(x) \equiv \int_0^{\infty} t^{x-1}e^{-t}\, dt \quad \text{for } x > 0.$$

The representation of $\mathcal{E}$ in terms of ${}_2F_1$ is

$$\mathcal{E}(r) \equiv \int_0^{\pi/2} (1 - r^2\sin^2\theta)^{1/2} d\theta = \frac{\pi}{2}\,{}_2F_1(-1/2, 1/2; 1; r^2)$$

and can be established by way of Euler's integral representation of ${}_2F_1$:

$${}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt$$

for $\gamma > \beta > 0$ and $z \in \mathbb{C}\backslash[1,\infty)$ (see [14]). By the earlier discussion involving $\hat{E}$, it follows that

$${}_2F_1(-1/2, 1/2; 1; 1-r^2) = \hat{E}(1,r),$$

which is a mean-value. As such, it is natural to seek relationships to other simpler means. M. Vuorinen [17] conjectured that ${}_2F_1(-1/2,1/2;1;1-r)^2$ could be sharply bounded below by the power mean of order 3/4. This was proven in [5]. Later, H. Alzer et al [3] found a sharp upper bound. Together, these results become

Theorem A. [3, 5] *For all* $r \in (0,1)$,

$$\mathcal{A}_\mu(1,r) \leq {}_2F_1(-1/2,1/2;1;1-r)^2 \leq \mathcal{A}_\lambda(1,r),$$

if $\mu \leq 3/4$ *(sharp*[1]*) and* $\lambda \geq \ln(\sqrt{2})/\ln(\pi/2)$ *(sharp).* The first inequality was proven in [5] and the second by Alzer et al in [3].

(Remark: Precursors of the inequalities in Theorem A date back to astronomers like Kepler who sought estimates of elliptical arc length. Some of these estimates took the form of power means. Ramanujan's interest in $\mathcal{E}$ led him to construct his own estimates (see [1]). Theorem A provides the best lower and upper power mean approximations to $\mathcal{E}$.)

An intriguing process for creating new means from old is that of compound iteration. From the arithmetic mean - geometric mean inequality, it can be shown that the recursively defined sequences given by $a_{n+1} = \mathcal{A}_1(a_n,b_n)$, $b_{n+1} = \mathcal{A}_0(a_n,b_n)$ (with $b_0 = y < x = a_0$) satisfy

$$b_n < b_{n+1} < a_{n+1} < a_n.$$

It follows that the two sequences have a common limit which is used to define the compound mean $\mathcal{A}_1 \otimes \mathcal{A}_0(x,y) \equiv \lim a_n = \lim b_n$, commonly referred to as the *arithmetic-geometric mean* $\mathcal{AG} \equiv \mathcal{A}_1 \otimes \mathcal{A}_0$. (See [6] for more on the historical development of $\mathcal{AG}$.) At this point a natural question arises: Does $\mathcal{AG}$ have a closed-form expression? A discovery by Gauss in 1799 provides an affirmative answer and results in the following remarkable identity (see [6, 14]):

$$\mathcal{AG}(1,r) = \frac{1}{{}_2F_1(1/2,1/2;1;1-r^2)}. \tag{3}$$

It follows from the definition of $\mathcal{AG}$ that $\mathcal{A}_0(x,y) < \mathcal{AG}(x,y) < \mathcal{A}_1(x,y)$ for all $x > y > 0$. However, $\mathcal{A}_1$ is not the best possible power mean upper bound for $\mathcal{AG}$. Since $a_2 = \left(\frac{x+y}{2} + \sqrt{xy}\right)/2 = \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 = \mathcal{A}_{1/2}(x,y)$, the above inequality can be improved to

$$\mathcal{A}_0(x,y) < \mathcal{AG}(x,y) < \mathcal{A}_{1/2}(x,y) \quad \text{for all } x > y > 0.$$

Vamanamurthy and Vuorinen [16] showed that the order 1/2 is *sharp* (i.e. best possible). This, together with the fact that $\mathcal{A}_\lambda$ is an increasing function

[1] (i.e., best possible)

of its order λ, yields

$$\mathcal{A}_\lambda(x,y) < \mathcal{AG}(x,y) < \mathcal{A}_\mu(x,y) \quad \text{for all } x > y > 0 \tag{4}$$

if and only if $\lambda \le 0$ and $\mu \ge 1/2$.
It is should also be noted that the complete elliptical integral of the first kind corresponds to this case of the hypergeometric function:

$$\mathcal{K}(r) \equiv \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{-1/2} d\theta = \frac{\pi}{2} {}_2F_1(1/2, 1/2, 1, r^2),$$

Sharp bounds for $\mathcal{K}$ follow from the work of verified by Vamanamurthy and Vuorinen in [16] discussed above for $\mathcal{AG}$; their result can be restated as

$$\mathcal{A}_\mu(1,r) \le \frac{1}{{}_2F_1(1/2, 1/2; 1; 1-r^2)} \le \mathcal{A}_\lambda(1,r), \tag{5}$$

if $\mu \le 0$ (sharp) and $\lambda \ge 1/2$ (sharp).
Results of the type in (5) and Theorem A naturally motivate a search for generalizations applicable when $(\pm a, b; c; \cdot)$ replaces $(\pm 1/2, 1/2; 1; \cdot)$. An initial challenge encountered in this context is to anticipate the form of the related power mean when working with the more general $(\pm a, b; c; \cdot)$. Guidance in this direction is provided by numerical evidence and by B.C. Carlson's work on the [10] *hypergeometric mean* and weighted power means given by

$$\mathcal{A}_\lambda(\omega; x, y) \equiv \Big(\omega\, x^\lambda + (1-\omega)\, y^\lambda\Big)^{1/\lambda} \quad (\lambda \ne 0)$$

and $\mathcal{A}_0(\omega; x, y) \equiv x^\omega y^{1-\omega}$, with weights $\omega, 1-\omega > 0$. (Note: Throughout the remaining discussion, the equally-weighted mean is implied if the weights are omitted: $\mathcal{A}_\lambda(x,y) = \mathcal{A}_\lambda(1/2; x, y)$.) In [10] (and the references therein), Carlson et al verified results that imply

$$\mathcal{A}_a(1 - b/c; 1, 1-r) \le {}_2F_1(-a, b; c; r)^{1/a} \quad \forall\, r \in (0,1), \tag{6}$$

if $1 \ge a$ and $c > b > 0$. Carlson [10] also showed that the inequality in (6) reverses if $a > 1$. Carlson's elegant proof uses a modification of Euler's integral representation and his work in this area is widely cited. Notice that, when $a = 1/2 = b = c/2$, the sharp power mean order is $3/4$ rather than a. This also motivates a search to replace the order a in (6) by the best possible order. Efforts to find *sharp power mean orders* and a natural generalization of Theorem A eventually resulted in the following:

Theorem B. [12] *Suppose* $1 \ge a,\ b > 0$ *and* $c > \max\{-a, b\}$. *If* $c \ge \max\{1 - 2a, 2b\}$, *then*

$$\mathcal{A}_\lambda(1 - b/c; 1, 1-r) \le {}_2F_1(-a, b; c; r)^{1/a} \quad \forall\, r \in (0,1)$$

if and only if $\lambda \le \frac{a+c}{1+c}$. *If* $c \le \min\{1-2a, 2b\}$, *then*

$$\mathcal{A}_\lambda(1-b/c; 1, 1-r) \ge {}_2F_1(-a, b; c; r)^{1/a}, \qquad \forall\, r \in (0,1)$$

if and only if $\lambda \ge \frac{a+c}{1+c}$. (The case $a = 0$ is treated as a limit and leads to the weighted *identric mean* discussed below.)

An interesting application[2] of Theorem B involves *hypergeometric analogues* of $\mathcal{AG}$ discussed by Borwein et al (see [7, 8] and the references therein). Using modular equations, Borwein et al [8] discovered compound means that can be expressed in closed form as

$$\mathcal{M} \otimes \mathcal{N}(1, r) = \frac{1}{{}_2F_1(1/2 - s, 1/2 + s; 1; 1 - r^p)^q}. \tag{7}$$

Motivated by a comparison with (3), compound means satisfying (7) are described as hypergeometric analogues of $\mathcal{AG}$. A problem of interest is to identify sharp inequalities similar to (4) for analogues of $\mathcal{AG}$. One approach is to use Theorem B and other basic series methods to arrive at

Theorem 1. [4] *Suppose* $0 < \alpha \le 1/2$ *and* $p > 0$. *Then for all* $r \in (0,1)$

$$\mathcal{A}_\lambda(\alpha; 1, r) < \frac{1}{{}_2F_1(\alpha, 1-\alpha; 1; 1-r^p)^{\frac{1}{\alpha p}}} < \mathcal{A}_\mu(\alpha; 1, r) \tag{8}$$

if $\lambda \le 0$ *(sharp) and* $\mu \ge p(1-\alpha)/2$ *(sharp).*

Obviously, such sharp inequalities are directly applicable to hypergeometric analogues of $\mathcal{AG}$. It is also worth noting that Theorem B may be applied to the more general class of *zero-balanced* hypergeometric functions of the form ${}_2F_1(a, b; a+b; \cdot)$ studied in [2, 4, 7, 8]. The next theorem in this section presents simultaneous sharp bounds for the zero-balanced class in terms of the power mean. The proof is obtained by naturally extending the verification of a similar result in [4] involving the hypergeometric analogues of the arithmetic-geometric mean.

Theorem 2. *Suppose* $1 + a \ge b \ge a > 0$. *Then for all* $r \in (0,1)$

$$\mathcal{A}_\lambda\left(\frac{a}{a+b}; 1, \frac{1}{(1-r)^a}\right) < {}_2F_1(a, b; a+b; r) < \mathcal{A}_\mu\left(\frac{a}{a+b}; 1, \frac{1}{(1-r)^a}\right) \tag{9}$$

if $\lambda \le \frac{-b}{a(1+a+b)}$ *(sharp) and* $\mu \ge 0$ *(sharp).*

Note that either lower or upper bounds are guaranteed by Theorem B under essentially complementary conditions on the parameters. As is seen in Theorem 1, it is also desirable to find simultaneous upper and lower bounds.

[2] This discussion regarding inequalities for analogues of $\mathcal{AG}$ is more fully developed in [4].

Using Alzer's [3] approach to the upper bound in Theorem A and numerical evidence, we make the following:

Conjecture I. *Suppose* $1 \geq a,\ c > b > 0$ *and* $c > b - a$.
If $c \geq \max\{1-2a, 2b\}$, *then for all* $r \in (0,1)$

$$\mathcal{A}_\mu(1-b/c;1,1-r) \leq {}_2F_1(-a,b;c;r)^{1/a} \leq \mathcal{A}_\lambda(1-b/c;1,1-r) \tag{10}$$

if $\mu \leq \frac{a+c}{1+c}$ *(sharp) and* $\lambda \geq \frac{a\ln(1-b/c)}{\ln\left(\frac{\Gamma(c+a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(c+a)}\right)}$ *(sharp).*
If $c \leq \min\{1-2a, 2b\}$, *then for all* $r \in (0,1)$

$$\mathcal{A}_\mu(1-b/c;1,1-r) \geq {}_2F_1(-a,b;c;r)^{1/a} \geq \mathcal{A}_\lambda(1-b/c;1,1-r) \tag{11}$$

if $\mu \geq \frac{a+c}{1+c}$ *(sharp) and* $\lambda \leq \frac{a\ln(1-b/c)}{\ln\left(\frac{\Gamma(c+a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(c+a)}\right)}$ *(sharp).*

Theorem B implies the first inequalities in (10) and (11). By Theorem A, the conjecture holds in the case that $a = b = c/2 = 1/2$. It is also interesting to note that, with a replaced by $-a$, the sharp value of $\lambda_c \equiv \frac{-a\ln(1-b/c)}{\ln\left(\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(c-a)}\right)}$ in Conjecture I has the property that $\lambda_c \to 0$ as c approaches $a+b$ from above. Thus Theorem 1 provides a verification of Conjecture I in the zero-balanced case in the following sense:

Corollary. *Suppose* $1 + a \geq b \geq a > 0$. *Then for all* $r \in (0,1)$

$$\mathcal{A}_\lambda\left(\frac{a}{a+b};1,1-r\right) < {}_2F_1(a,b;a+b;r)^{-1/a} < \mathcal{A}_\mu\left(\frac{a}{a+b};1,1-r\right)$$

if $\lambda \leq 0$ *(sharp) and* $\mu \geq \frac{b}{(1+a+b)}$ *(sharp).*

Before moving to another application of these inequalities, we state the following:

Conjecture II. *If* $a > 1$ *and* $b > 0$, *then*

$$\mathcal{A}_\mu(1/2;1,1-r) \leq {}_2F_1(-a,b;2b;r)^{1/a} \leq \mathcal{A}_\lambda(1/2;1,1-r) \quad \forall r \in (0,1)$$

if $\lambda \geq \frac{a+2b}{1+2b}$ *(sharp) and* $\mu \leq \frac{a\ln(2)}{\ln\left(\frac{\Gamma(b)\Gamma(2b+a)}{\Gamma(2b)\Gamma(b+a)}\right)}$ *(sharp).*

As more evidence for these conjectures, we have the following propositions that address the case that $b = 1$ and $c = 2$.

Proposition 1. [15] *Suppose* $a \in (-1,1)$. *If* $a > -1/2$, *then for all* $r \in (0,1)$

$$\mathcal{A}_\mu(1/2;1,1-r) \leq {}_2F_1(-a,1;2;r)^{1/a} \leq \mathcal{A}_\lambda(1/2;1,1-r) \tag{12}$$

if $\mu \leq \frac{a+2}{3}$ *(sharp) and* $\lambda \geq \frac{a\ln(2)}{\ln(1+a)}$ *(sharp). If* $a < -1/2$, *then for all* $r \in (0,1)$

$$\mathcal{A}_\mu(1/2;1,1-r) \geq {}_2F_1(-a,1;2;r)^{1/a} \geq \mathcal{A}_\lambda(1/2;1,1-r) \tag{13}$$

if $\mu \geq \frac{a+2}{3}$ *(sharp) and* $\lambda \leq \frac{a \ln(2)}{\ln(1+a)}$ *(sharp).*

Proposition 2. [15] *If* $a > 1$*, then*

$$\mathcal{A}_\mu(1/2; 1, 1-r) \leq {}_2F_1(-a, 1; 2; r)^{1/a} \leq \mathcal{A}_\lambda(1/2; 1, 1-r) \quad \forall r \in (0, 1)$$

if $\lambda \geq \frac{a+2}{3}$ *(sharp) and* $\mu \leq \frac{a \ln(2)}{\ln(1+a)}$ *(sharp).*

Proofs for both of these propositions are developed in [15] by applying Páles' Theorem [11] discussed below. This process involves the Stolarsky mean which is defined for $ab(a-b) \neq 0$ as

$$\mathcal{D}_{a,b}(x, y) \equiv \left(\frac{b(x^a - y^a)}{a(x^b - y^b)} \right)^{1/(a-b)}.$$

The connection between the work by Páles and the hypergeometric function becomes clear by noting that

$$\begin{aligned} {}_2F_1(-a, 1; 2; r)^{1/a} &= \left(\sum_{n=0}^{\infty} \frac{(1-\sigma)_n (1)_n}{(2)_n n!} r^n \right)^{1/(\sigma-1)} \quad \text{(substituting } \sigma - 1 \text{ for } a\text{)} \\ &= \left(\frac{1 - (1-r)^\sigma}{\sigma r} \right)^{1/(\sigma-1)} \end{aligned}$$

Thus

$${}_2F_1(-a, 1; 2; r)^{1/a} = \mathcal{D}_{1+a,1}(1, 1-r).$$

The power mean can also be written as a specific form of the Stolarsky mean:

$$\begin{aligned} \mathcal{A}_\lambda(1/2; x, y) &= \left(\frac{x^\lambda + y^\lambda}{2} \right)^{1/\lambda} \\ &= \left(\left(\frac{\lambda}{2\lambda} \right) \left(\frac{x^{2\lambda} - y^{2\lambda}}{x^\lambda - y^\lambda} \right) \right)^{1/\lambda} \\ &= D_{2\lambda,\lambda}(x, y) \end{aligned} \tag{14}$$

Using these representations and the following theorem, it is possible to obtain proofs of the above propositions.

Páles' Theorem. [11] *Let* $a,\ b,\ c,\ d \in \mathbb{R}$*. Then the comparison inequality*

$$\mathcal{D}_{(a,b)}(x, y) \leq \mathcal{D}_{(c,d)}(x, y)$$

holds true for all $x,\ y > 0$ *if and only if* $a + b \leq c + d$ *and*

$$\begin{array}{ll} \mathcal{L}(a, b) \leq \mathcal{L}(c, d) & \text{if } 0 \leq \min(a,\ b,\ c,\ d) \\ \dfrac{|a| - |b|}{a - b} \leq \dfrac{|c| - |d|}{c - d} & \text{if } \min(a,\ b,\ c,\ d) < 0 < \max(a,\ b,\ c,\ d) \\ -\mathcal{L}(-a, -b) \leq -\mathcal{L}(-c, -d) & \text{if } \max(a,\ b,\ c,\ d) \leq 0. \end{array}$$

The final related theorem to be presented involves generalizations of $\mathcal{L}$ and $\mathcal{I}$ defined using the bivariate form of Carlson's [10] hypergeometric mean

$$\mathcal{H}_a(\omega; c; x, y) \equiv x \cdot [{}_2F_1(-a, c\,\omega; c; 1 - y/x)]^{\frac{1}{a}}$$

with the parameter $c > 0$ and weights ω, $1-\omega > 0$. (Note that Theorems A and B can be stated in terms of $\mathcal{H}_a$.) This leads to the *weighted logarithmic mean* $\hat{\mathcal{L}}$ which is defined as

$$\hat{\mathcal{L}}(\omega; c; x, y) \equiv \mathcal{H}_{-1}(\omega; c; x, y).$$

It can be shown that $\hat{\mathcal{L}}(1/2; 2; x, y) = \mathcal{L}(x, y)$. Similarly, the *weighted identric mean* $\hat{\mathcal{I}}$ is given by

$$\hat{\mathcal{I}}(\omega; c; x, y) \equiv \mathcal{H}_0(\omega; c; x, y) \equiv \lim_{a \to 0} \mathcal{H}_a(\omega; c; x, y).$$

It turns out that $\hat{\mathcal{I}}(1/2; 2; x, y) = \mathcal{I}(x, y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{1/(x-y)}$. We have the following recent result involving the hypergeometric mean that generalizes the well-known inequalities $\mathcal{L} \leq \mathcal{A}_{1/3}$ and $\mathcal{A}_{2/3} \leq \mathcal{I}$ for the equally-weighted case. (See [9] for a thorough discussion of other related means and inequalities.)

Theorem C. [13] *Suppose $x > y > 0$ and $c > b > 0$.*
If $\max\{1, 2b\} \leq c$, then the weighted identric mean $\hat{\mathcal{I}}$ with $\omega = 1 - b/c$ satisfies

$$\mathcal{A}_{\frac{c}{c+1}}(\omega; x, y) \leq \hat{\mathcal{I}}(b, c; x, y).$$

If $1 < c \leq \min\{3, 2b\}$, then the weighted logarithmic mean $\hat{\mathcal{L}}$ with $\omega = 1 - b/c$ satisfies

$$\hat{\mathcal{L}}(b, c; x, y) \leq \mathcal{A}_{\frac{c-1}{c+1}}(\omega; x, y).$$

Moreover, the power mean orders $c/(c+1)$ and $(c-1)/(c+1)$ are sharp.
We close by noting that the sharp power mean bounds for the corresponding multivariate identric and logarithmic means (see [9]) remains as an open problem.

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